

Phase diagram of two-color QCD matter at finite baryon and axial isospin densities*

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Abstract: We study the two-color QCD matter with two fundamental quark flavors using the chiral perturbation theory and the Nambu-Jona-Lasinio (NJL) model. The effective Lagrangian is derived in terms of mesons and baryons, *i.e.* diquarks. The low lying excitations lie in the extended $SU(4)$ flavor symmetry space. We compute the leading order terms of the Lagrangian as a function of the baryon and axial isospin densities. After numerically solving the gap equations in the two-color NJL model, the phase diagram is obtained in the $\mu - \nu_5$ plane.

Keywords: two-color QCD, diquarks, chiral perturbation theory, axial isospin density, NJL model

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1 Introduction

The color superconductivity in Quantum Chromodynamics (QCD) matter at high baryon density, such as in the core of a neutron star, has been extensively studied, see e.g. [1] for a review and references therein. Several studies have shown that the consequences of the large superconducting gap should be observed in astrophysics or heavy-ion collision experiments. However, this has not been definitively confirmed by the first principles calculations due to the sign problem in the lattice Monte Carlo simulations in dense QCD matter [2, 3]. It has therefore been proposed to understand the quark matter at finite baryon density in simpler QCD-like theories. For example, in the $N_c = 2$ gauge theory, the fermion determinant is guaranteed to be real for nonzero chemical potential due to the additional anti-unitary symmetry, which allows a discrete evaluation of the fundamental and adjoint quarks [4-15]. On the other hand, the flavor space could be extended from the usual $SU(N_f)$ to the larger $SU(2N_f)$ [4, 7, 8]. The two-color QCD framework (QC₂D) is somewhat different from the full three-color gauge theory in the sense that the baryons are bosons rather than fermions. Nevertheless, this approach sheds light on the mechanism of phase transitions, the formation of diquark condensate and critical phenomena, in realistic QCD matter.

The investigation of the QCD phase diagram for finite baryon and nonzero isospin densities has attracted

considerable attention [8, 16-19]. The main reason is that the dense hadronic matter is isotopically asymmetric in heavy-ion collision experiments [20, 21]. The interior of compact stars is also expected to be isospin asymmetric [22]. A large class of effective theories, such as the chiral perturbation theory [8, 16], the Nambu-Jona-Lasinio (NJL) type models [23-27], linear sigma models [28], the Polyakov-Quark-Meson models [29, 30], the Dyson-Schwinger equations [31] and functional renormalization group methods [32, 33], have been used to explore the alignment of the vacuum expectation value (VEV) and the matter phase structure.

The study of QCD matter in external electromagnetic (EM) environment [34-40] has become of interest since the charge separation along the magnetic field was observed in non-central heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) [41]. A background EM field exists in various physical systems, such as in compact stars [42], the early Universe [43] and at RHIC and the Large Hadron Collider [41, 44, 45]. The EM field excites charged particles in both the hadronic and quark-gluon plasma (QGP) phases, offering a unique opportunity for exploring the QCD vacuum structure and the thermodynamics of strongly interacting matter. Several approaches have been used to understand the phenomena of magnetic catalysis at low temperature, and of the inverse magnetic catalysis near T_c in magnetized quark matter [46-50]. It was suggested that the QCD vacuum becomes

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a superconductor due to charged ρ condensation [51-55]. Recently, the EM chiral anomaly has triggered new interest in investigating the QCD phase diagram in a (anti)parallel EM environment, *i.e.* in (anti)parallel electric and magnetic fields [56-58]. In [56], the authors found that the pion superfluidity is energetically favored when the strength of EM fields is increased. Indeed, the chiral anomaly is closely linked to various quantum phenomena in EM fields, such as the chiral magnetic effect [59], chiral magnetic waves [60] and chiral electrodynamics [61].

As a consequence, there is a good reason to study the behavior of diquarks in electromagnetized dense baryon matter, as the EM field acts on all charged collective modes, and the diquarks can be treated as elementary states of QCD matter at high baryon density [7, 9]. If only the QCD interactions are considered, the axial isospin current is anomaly free. The EM chiral anomaly arises when quarks couple to the electromagnetic field. The corresponding current is given by

$$\partial_\mu J_5^{\mu 3} = -\frac{e^2}{16\pi^2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \cdot \text{tr}[\tau_3 Q^2], \quad (1)$$

where Q is the matrix of quark electric charges, τ_3 is the Pauli matrix in flavor space and F is the field strength. A related process is the decay of a neutral pion into two photons [62-64]. In order to qualitatively study the charge asymmetry in a strong EM field, we introduce an axial isospin chemical potential ν_5 , and $\nu_5 \bar{\psi} \gamma_0 \gamma_5 \tau_3 \psi$ is included in the quark sector of the Lagrangian [65, 66]. A similar fermionic part of the Lagrangian, in the parameter space of the isospin chemical potential instead of ν_5 , was studied in [67] with twisted mass Wilson fermions.

In order to determine the properties of diquarks in EM fields, we compute in this work the QCD phase diagram in the plane of baryon and axial isospin chemical potentials in two-color QCD, which consists of the two-color gauge group with two Dirac flavors in the fundamental representation. This model dynamically generates quasi-particles in terms of sigmas, pions and baryons (diquarks). The details of the model are given in Sec. 2. In Sec. 3, we derive the underlying effective Lagrangian. Using the static low-energy effective \mathcal{L} , we identify the breaking patterns of the global symmetries. The NJL model is solved numerically and the results are shown in Sec. 4. The conclusions are presented in the final section.

2 Lagrangian of the two-color QCD

We start with the following NJL Lagrangian for the QC₂D model, where the effective four-fermion interactions are included in a gluon inspired manner; more details can be found in [11]. The Lagrangian reads

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} \left(i \gamma^\mu \partial_\mu + \mu \gamma_0 + \nu_5 \gamma_0 \gamma_5 \tau_3 - m_0 \right) \psi + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{qq} + (\text{color triplet terms}), \quad (2)$$

$$\mathcal{L}_{\bar{q}q} = \frac{G}{2} \left[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \vec{\tau} \psi)^2 \right], \quad (3)$$

$$\mathcal{L}_{qq} = \frac{H}{2} (\bar{\psi} i \gamma_5 \tau_2 t_2 C \bar{\psi}^T) (\psi^T C i \gamma_5 \tau_2 t_2 \psi) - \frac{H}{4} (\bar{\psi} \gamma_3 \tau_1 t_2 C \bar{\psi}^T) (\psi^T C \gamma_3 \tau_1 t_2 \psi), \quad (4)$$

where $C = i \gamma_0 \gamma_2$ is the charge conjugation operator, m_0 is the current quark mass, and τ_i and t_i are the Pauli matrices in flavor and color spaces, respectively. τ_2, t_2 are the anti-symmetric matrices. The two coupling constants G and H are connected by the Fierz transformation in color space. In particular, $G = H$ for $N_c = 2$.

Since $\nu_5 = \mu_L^u - \mu_R^u = \mu_R^d - \mu_L^d$, where the left-handed u quark is on the same Fermi surface as the right-handed d quark, while the baryon density of the right-handed u quark is equal to the left-handed d quark, the corresponding LR bound states composed of two quarks on equal Fermi surfaces are $\psi_L^u \psi_R^d \pm \psi_R^u \psi_L^d$. Focusing on the anti-symmetric color sector, where the attraction is stronger, the Dirac structure of the total antisymmetric LR diquark $\psi_L^u \psi_R^d - \psi_R^u \psi_L^d$ is in the repulsive ($C \gamma_0 \gamma_5$) channel with zero spin and isospin, which cannot form a bound particle. On the other hand, the flavor symmetric state, $\psi_L^u \psi_R^d + \psi_R^u \psi_L^d$ is attractive with Dirac operator ($C \gamma_3$). Symmetric flavor diquark pairing patterns have been explored in the same-or one-flavor color superconductors [68-74]. It was suggested that, except for the most attractive scalar diquark ($C \gamma_5$), the axial spin diquark pairs are formed. Hence, for finite ν_5 , the system prefers the polar phase ($s = 1, s_z = 0$) for symmetric spin, and the τ_1 channel for symmetric flavor, to satisfy the Pauli exclusion principle.

In [56, 75], the authors showed that the QCD vacuum makes a chiral rotation from the scalar σ condensate to the pseudo-scalar π_i condensates as a result of the EM chiral anomaly. Therefore, based on these considerations, we apply in this work the Hubbard-Stratonovich transformation and introduce three kinds of auxiliary meson and diquark fields: $\pi_3 = -\frac{G}{2} \langle \bar{\psi} i \gamma_5 \tau_3 \psi \rangle$ represents the neutral pseudo-scalar, $\Delta = -\frac{H}{2} \langle \psi^T i \gamma_5 \tau_2 t_2 C \psi \rangle$ and $\Delta^* = -\frac{H}{2} \langle \bar{\psi} i \gamma_5 \tau_2 t_2 C \bar{\psi}^T \rangle$ denote the complex scalar diquarks, and $d = -\frac{H}{4} \langle \psi^T \gamma_3 \tau_1 t_2 C \psi \rangle$ and $d^* = -\frac{H}{4} \langle \bar{\psi} \gamma_3 \tau_1 t_2 C \bar{\psi}^T \rangle$ are the complex axial vector diquarks. In terms of the Nambu-Gorkov bispinors

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} q \\ C \bar{q}^T \end{pmatrix}, \quad \bar{\psi} = \frac{1}{\sqrt{2}} (\bar{q}, q^T C), \quad (5)$$

the effective thermodynamical potential is written as

$$\Omega_{\text{eff}} = \frac{1}{2} \bar{\psi} \mathcal{S}^{-1}(p; \pi_3, \Delta, d) \psi - \frac{\pi_3^2 + |\Delta|^2 + |d|^2}{4G}. \quad (6)$$

Here, the inverse propagator of the fermion is

$$S^{-1}(p) = \begin{pmatrix} \not{p} - M + \gamma_0 \mu + \gamma_0 \gamma_5 \tau_3 \nu_5 & \tilde{\Delta} \\ -\tilde{\Delta}^\dagger & \not{p} - M^T - \gamma_0 \mu + \gamma_0 \gamma_5 \tau_3 \nu_5 \end{pmatrix}, \quad (7)$$

where $M = (m_0 - \sigma)\tau_0 - i\gamma_5 \tau_3 \pi_3$ and $\tilde{\Delta} = \gamma_5 \tau_2 \Delta + \gamma_3 \tau_1 d$. The color index has been omitted since it becomes trivial in the calculations below.

3 Mesons and diquarks in chiral perturbation theory

In this section, we briefly explain the behavior of π_3 , Δ and d given the external conditions in the framework of the chiral perturbation theory.

The fundamental representation of the group $SU(2)$ is pseudo-real and is isomorphic to its complex conjugate representation with the isometry given by $S = i\sigma_2$ [4, 7, 8]. In other words, charge conjugation of the gauge fields when $N_c = 2$ is equivalent to a constant gauge transformation of S . Therefore, left- and right-handed spinors get interchanged under the anti-unitary symmetry $T = CSK$, where C is the charge conjugation matrix in spinor space and K is complex conjugation. One may write the spinors as

$$\Psi = \begin{pmatrix} \psi_L \\ \tilde{\psi}_R \end{pmatrix}, \quad \Psi^\dagger = (\psi_L^\dagger, \tilde{\psi}_R^\dagger), \quad (8)$$

where $\tilde{\psi}_R = T\psi_R = -i\sigma_2 S\psi_R^*$ and $-i\sigma_2$ is the C matrix for the right spinor. This gauge theory has a larger flavor symmetry $SU(2N_f)$. In the extended flavor space, Ψ manifests its flavor symmetry and chiral component at the same time, establishing a connection between quarks and anti-quarks [4, 7, 8]. As a consequence, color singlet baryons are composed of two quarks, and scalar diquarks become degenerate with pseudo-mesons (pions).

Using the above expression, the standard kinetic part of the Euclidean QC₂D Lagrangian can be written as [14]

$$\mathcal{L}_{\text{kin}} = \Psi^\dagger i\sigma^\mu D_\mu \Psi, \quad (9)$$

where σ^μ is the Hermitian gamma matrix $\sigma^\mu = (-i, \vec{\sigma})$, and D_μ is the covariant derivative $D_\mu = \partial_\mu + iA_\mu^a T_a$. The coupling constant is absorbed in the gauge fields. Similarly, the quark mass term becomes [14]

$$\mathcal{L}_{\text{mass}} = \frac{m_0}{2} (\Psi^T i\sigma_2 S E_4 \Psi - \Psi^{*T} i\sigma_2 S E_4 \Psi^*), \quad (10)$$

where the symplectic matrix E_4 is

$$E_4 = \begin{pmatrix} 0 & \tau_0 \\ -\tau_0 & 0 \end{pmatrix}. \quad (11)$$

The group $SU(4)$ has 15 generators. In the block representation, the 10 symmetric elements are [76, 77]

$$S_a = \frac{1}{2\sqrt{2}} \begin{pmatrix} \tau_a & 0 \\ 0 & -\tau_a^T \end{pmatrix}, \quad \text{for } a = 0, 1, 2, 3; \quad (12)$$

$$S_a = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & B_a \\ B_a^\dagger & 0 \end{pmatrix}, \quad \text{for } a = 4, \dots, 9, \quad (13)$$

with $B_{(4,5)} = i^{(0,1)}\tau_0$, $B_{(6,7)} = i^{(0,1)}\tau_3$ and $B_{(8,9)} = i^{(0,1)}\tau_1$. The remaining five anti-symmetric generators are:

$$X_i = \frac{1}{2\sqrt{2}} \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i^T \end{pmatrix}, \quad \text{for } i = 1, 2, 3; \quad (14)$$

$$X_i = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & D_i \\ D_i^\dagger & 0 \end{pmatrix}, \quad \text{for } i = 4, 5, \quad (15)$$

with $D_{(4,5)} = i^{(0,1)}\tau_2$.

$SU(4)$ explicitly (dynamically) reduces to the symplectic group $Sp(2)$ as a result of the current (constituent) quark mass. The generators of $Sp(2)$ obey the relation $S_a^T E_4 + E_4 S_a = 0$, with the mass operator E_4 , which clearly indicates that the bare mass term is only invariant under the subgroup of $Sp(2)$. Thus, E_4 and X_i , the elements of the coset subgroup $SU(4)/Sp(2)$, form a six-dimensional vector of chiral order parameters under the quark bilinears $(\Psi^T \vec{\phi} \Psi + \text{h.c.})$. Five almost Goldstone bosons $\vec{\pi}, \Delta$ and Δ^* are created. Note that here $\vec{\phi} = (E_4, iE_4 X_i, X_j)$ for $i = 1, 2, 3$ and $j = 4, 5$, where E_4 is the generator of scalar meson σ , $iE_4 X_{(1,2,3)}$ represents the pseudo-scalar triplet pions $\vec{\pi}$, and $X_{(4,5)}$ denote the scalar diquark and anti-diquark Δ and Δ^* .

Employing the above representation, the conventional baryon Lagrangian term $\mu \bar{\psi} \gamma_0 \psi$ is rewritten in the $SU(4)$ space as $\mu \Psi^\dagger B_0 \Psi$ [14], where

$$B_0 = -\gamma_0 E_4 = \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}. \quad (16)$$

Similarly, the isospin current $\nu \bar{\psi} \gamma_0 \tau_3 \psi$ is written as $\nu \Psi^\dagger I_0 \Psi$ with $I_0 = \text{Diag}(\tau_3, -\tau_3)$. The term with axial isospin density $\nu_5 \bar{\psi} \gamma_0 \gamma_5 \tau_3 \psi$ is expressed as $\nu_5 \Psi^\dagger I_5 \Psi$ with $I_5 = \gamma_5 I_0 = \text{Diag}(\tau_3, \tau_3)$.

For finite isospin density, applying the gauge transformation [7, 8]

$$\Psi \rightarrow V \Psi, \quad V = \exp(i\theta^i X_i) \quad \text{for } X_i \in SU(4)/Sp(2), \quad (17)$$

one has $[X_i, I_0] \neq 0$ for $i = 1, 2$. This means, as is well known, that the isospin baryon current is embedded in the charged mesons π_\pm to compensate the gauge transformation. Replacing the derivatives by covariant derivatives, one gets the leading order terms of the Lagrangian in the chiral perturbation theory (χ PT) [7, 8]

$$\mathcal{L}_{\chi\text{PT}} = \frac{f_\pi^2}{4} \text{Tr}(D_\mu \Sigma^\dagger)(D_\mu \Sigma) - c \text{Tr}(\Sigma^\dagger + \Sigma), \quad (18)$$

where the unitary matrix Σ consists of the fluctuations of the symmetry breaking order parameters with respect to the extended $SU(4)$ flavor group, and the second term is induced by the quark mass. As shown in [8], the long covariant derivatives can be written explicitly as:

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \Sigma + (I_\mu \Sigma + \Sigma I_\mu^T) \\ D_\mu \Sigma^\dagger &= \partial_\mu \Sigma^\dagger - (\Sigma^\dagger I_\mu + I_\mu^T \Sigma^\dagger). \end{aligned} \quad (19)$$

After expansion, two non-derivative terms appear in a linear combination of

$$\text{Tr}(I_\mu^T \Sigma^\dagger I_\mu \Sigma), \quad \text{and} \quad \text{Tr}(I_\mu I_\mu), \quad (20)$$

because the trace is invariant under permutation. The second term, which is constant, is discarded. Therefore, ignoring the kinetic and mass terms, to demonstrate that the chiral orientation towards the charged pions at finite isospin density are energetically favored, the remaining static Lagrangian in [8] is expressed as

$$\mathcal{L}_{\chi\text{PT}} = \mathcal{L}_\pi \sim v^2 \text{Tr}(I_0^T \phi_i^\dagger I_0 \phi_i) = v^2 \text{Diag}(\tau_3 \tau_i^T \tau_3 \tau_i^T, \tau_3 \tau_i \tau_3 \tau_i), \quad (21)$$

where $\phi_i = E_\Delta X_i$, characterizing the bilinear meson field $\psi^\dagger \psi$. When the fluctuations of the vacuum expectation value of Σ are neglected, the above term becomes negative for $\tau_i = \tau_{1,2}$, which are correlated with $\phi_i \sim X_{1,2}$, indicating that the charged pion condensate is formed for $\mu_l > m_\pi/2$, as expected.

At finite baryon chemical potential μ , the meson fields are unaffected by the baryon current since $[X_i, B_0] = 0$ for $i = 1, 2, 3$. Not surprisingly, the (anti)diquark generators $X_{4,5}$ do not commute with B_0 . The covariant requirement modifies the scalar (anti)diquark fields and the effective Lagrangian term is [7]

$$\begin{aligned} \mathcal{L}_{\chi\text{PT}} &= \mathcal{L}_{\Delta, \Delta^*} \sim \mu^2 \text{Tr}(B_0^T \Phi_i^\dagger B_0 \Phi_i) \\ &= -\mu^2 \text{Diag}(\tau_0 t_i \tau_0 t_i^\dagger, \tau_0 t_i^\dagger \tau_0 t_i). \end{aligned} \quad (22)$$

Here, $t_i = i^{(0,1)} \tau_2$ because of the complex diquarks Δ and Δ^* . In the conventional $SU(2)$ space, they are composed of two (anti-)quarks, $\psi^T \psi$.

When the axial isospin chemical potential $v_5 \neq 0$, excluding the commuted generator X_3 with respect to I_5 , all other antisymmetric generators appear in the covariant part of the Lagrangian to compensate the gauge transformation of the axial isospin current. The associated Lagrangian terms in mesons space are

$$\begin{aligned} \mathcal{L}_{\chi\text{PT}} &= \mathcal{L}_\pi \sim v_5^2 \text{Tr}(I_5^T \phi_i^\dagger I_5 \phi_i) \\ &= -v_5^2 \text{Diag}(\tau_3 \tau_i^T \tau_3 \tau_i^T, \tau_3 \tau_i \tau_3 \tau_i), \end{aligned} \quad (23)$$

which is *positive* and increases the effective potential for $\phi_i \sim X_{1,2}$. This is the reason why in this work π_3 is chosen to characterize the meson type of the chiral condensate at finite v_5 .

Moreover, in the high baryon density limit, where flavor symmetric diquarks spontaneously form a condensate, the remaining $Sp(2)$ group continues to break to a reduced symmetry. It is expected that $Sp(2)$ reduces to $O(4)$ [76, 77]. We find that $[S_a, I_5] \neq 0$ for $a = 1, 2, 8, 9$. The $O(4)$ group is left due to the six commuting elements of S_a . In addition, the number of would be Goldstone bo-

sons increases from 5 to 9. Two of them are the flavor triplet scalar mesons a_0^\pm . The other two are the axial (anti)diquarks d, d^* . As the goal of this work is to explore the region of cold high-density quark matter where the diquark pairing pattern and its behavior is more interesting than mesons, the possible appearance of new meson condensates is beyond the scope of this work. The static Lagrangian term of the diquark is modified to

$$\begin{aligned} \mathcal{L}_{\chi\text{PT}} &= \mathcal{L}_{\Delta, d} \sim v_5^2 \text{Tr}(I_5^T \Phi_i^\dagger I_5 \Phi_i) \\ &= v_5^2 \text{Diag}(\tau_3 t_i \tau_3 t_i^\dagger, \tau_3 t_i^\dagger \tau_3 t_i), \end{aligned} \quad (24)$$

which is negative for $t_i = i^{(0,1)} \tau_{1,2}$. This means that Φ_i are favored as $X_{4,5}$ or $S_{8,9}$. The former two correspond to the scalar diquark Δ and anti-diquark Δ^* , ($C\gamma_5 \tau_2 t_2$), while the latter two are the axial vector diquark d and anti-diquark d^* . In the two-color space, the color singlet baryon, *i.e.* diquark, is composed in the anti-symmetric color channel. Thus, the flavor symmetric diquarks d and d^* have to be symmetric in spin space, ($C\gamma_3 \tau_1 t_2$), to maintain the total antisymmetry [68-74]. It should be noted that the choice of the τ_1 flavor space is consistent with the generator of $S_{8,9}$. In summary, we have shown that three kinds of condensates, $\pi_3, \Delta(\Delta^*), d(d^*)$, appear at the leading order of the low energy effective theory for both the finite chemical and axial isospin chemical potentials.

4 Numerical results

Considering the antisymmetric color channel alone for $N_c = 2$, a diquark is either a spin-, isospin-singlet (LL or RR quark states), or a constituent of the spin-, isospin-triplets (LR quark states), whose color space does not mix with the spin direction. Note that if three colors are available, the spins can be locked to the colors in a linear manner, where $\lambda_{A=2,5,7}$ correlate with the spatial directions of ($C\gamma_{i=1,2,3}$). An unbroken global $SO(3)$ mixture remains and the gap is isotropic, which lowers the free energy as shown in [72].

We use the single Dirac component ($C\gamma_3$) to denote the spin one diquarks. In what follows, we assume that $\Delta = \Delta^*$, $d = d^*$ are real quantities, as is done in most works, and apply the mean-field (Hartree) approximation. It is convenient to write the Lagrangian in the Nambu-Gorkov spinors space, as was done in Sec. 2. The thermodynamical potential is derived as

$$\begin{aligned} \Omega(T, \mu, v_5) &= \frac{T}{2V} \ln \det \mathcal{S}^{-1}(i\omega_n, p; \pi_3, \Delta, d) + \frac{\pi_3^2 + \Delta^2 + d^2}{4G} \\ &= -T \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \text{Tr} \ln \left(\frac{1}{T} \mathcal{S}^{-1}(i\omega_n, p; \pi_3, \Delta, d) \right) \\ &\quad + \frac{\pi_3^2 + \Delta^2 + d^2}{4G}, \end{aligned} \quad (25)$$

where the sum is over fermionic Matsubara frequencies $\omega_n = (2n + 1)\pi T$. Taking into account the color, flavor and Dirac structure, \mathcal{S}^{-1} is a 16×16 matrix.

From the above expression, one obtains three gap equations by minimizing the thermodynamic potential with respect to the mean values of the three meson and diquark fields [11, 78]

$$\frac{\partial \Omega}{\partial \pi_3} = 0, \quad \frac{\partial \Omega}{\partial \Delta} = 0, \quad \frac{\partial \Omega}{\partial d} = 0. \quad (26)$$

With the identity $\text{Tr} \ln A = \ln \det A$, the trace in Eq. (25) is evaluated as

$$\frac{1}{2} \text{Tr} \ln \left(\frac{1}{T} \mathcal{S}^{-1}(i\omega_n, p; \pi_3, \Delta, d) \right) = \sum_{i=1}^8 \ln \left(\frac{\omega_n^2 + E_i^2}{T^2} \right). \quad (27)$$

In the chiral limit, $m_0 \rightarrow 0$, and assuming $\pi_3 = \Delta = 0$, the doubly degenerate energy dispersions of the collective modes are

$$E_i(\vec{p}) = \pm \sqrt{d^2 + \vec{p}^2 + (\mu \pm v_5)^2} \pm 2 \sqrt{d^2 p_z^2 + \vec{p}^2 (\mu \pm v_5)^2}. \quad (28)$$

$E_i(\vec{p})$ are guaranteed to be real since $\vec{p}^2 \geq p_z^2$, and the result agrees with the early studies for $v_5 = 0$ [72]. Unfortunately, with nonzero quark mass, or for a pion superfluid, or when the scalar diquark condensate is included, the formulas for the energy become very complicated. Therefore, we proceed with numerical calculations in this work.

Employing the two-color NJL model, the thermodynamical potential becomes

$$\Omega(T, \mu, v_5) = - \sum_{i=1}^8 \int \frac{d^3 p}{(2\pi)^3} \left[E_i(\vec{p}) \theta(\Lambda^2 - \vec{p}^2) + 2T \ln \left(1 + e^{-\frac{E_i(\vec{p})}{T}} \right) \right] + \frac{\pi_3^2 + \Delta^2 + d^2}{4G}, \quad (29)$$

where θ is the step function. In the calculations, the parameter set from Ref. [11] is used, where $\Lambda = 0.78$ GeV, $G = H = 10.3$ GeV $^{-2}$ and $m_0 = 4.5$ MeV. Plugging the numerical results of $E_i(\vec{p})$ into Eq. (29), we investigate the phase diagram of the two-color QCD with two fundamental quarks at zero temperature, focusing in particular on the nature of diquarks. As was shown above, Δ and $\vec{\pi}$ are in the same coset of $SU(4)/Sp(2)$, which means that the mass of Δ is the same as that of pions in vacuum. Without v_5 , the scalar diquark condensate appears when the chemical potential μ exceeds half of $m_\Delta = m_\pi \sim 0.14$ GeV. Therefore, to focus on the behavior of diquarks, the presented phase structure starts from 0.07 GeV for μ . The results are plotted in the $\mu - v_5$ plane in Fig. 1. It may be noted that the plot shows four phases, a mixed state, pion superfluid (π_3), scalar (Δ) and axial vector (d) diquark condensates. The mixed state hybridizes π_3 and Δ . This is easy to understand as both charac-

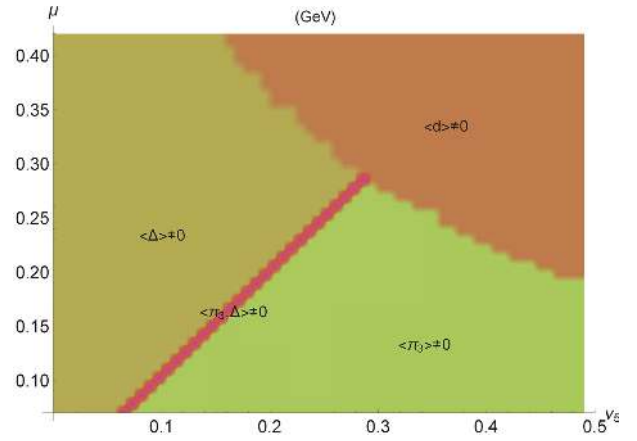


Fig. 1. (color online) Phase diagram of the two-color QCD with two fundamental quarks in the $\mu - v_5$ plane at zero temperature.

terize the chiral symmetry breaking, like the mass term, which is a feature of QC $_2$ D theory and differs from the three-color QCD. On the other hand, a nonzero VEV of d appears for high values of μ and v_5 . It derives from the symmetry breaking $Sp(2) \rightarrow O(4)$, so that the spin one diquark is a competitor of the chiral condensate, and hence never blends with either π_3 or Δ .

The VEVs of the three condensates are shown as a function of the axial isospin chemical potential and/or temperature in Figs. 2, 3 and 4. These plots demonstrate that the gap of the spin one color superconductor is smaller than the chiral condensate, as revealed in the early studies of the $N_c = 3$ color superconductivity [1, 72]. We observe that as the temperature increases, all VEVs of the fields decrease, as shown in Fig. 2. The system undergoes a phase transition to a quark-gluon plasma at high temperatures. The order parameters as function of v_5 for different values of chemical potential and temperature are also shown. The pion superfluid vanishes at high temperature ($T = 150$ MeV) and the chiral symmetry is only

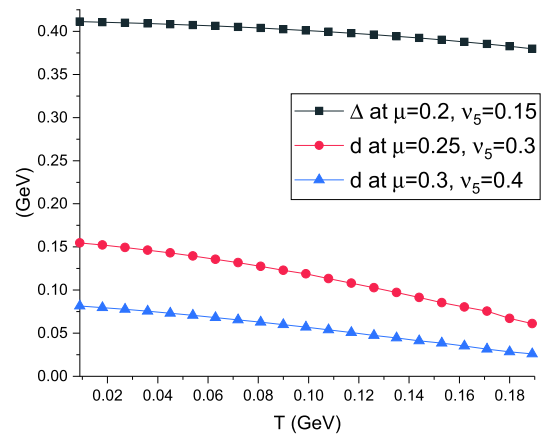


Fig. 2. (color online) Scalar diquark condensate Δ and axial vector diquark d as a function of T for a given μ and v_5 . The units of μ and v_5 are GeV.

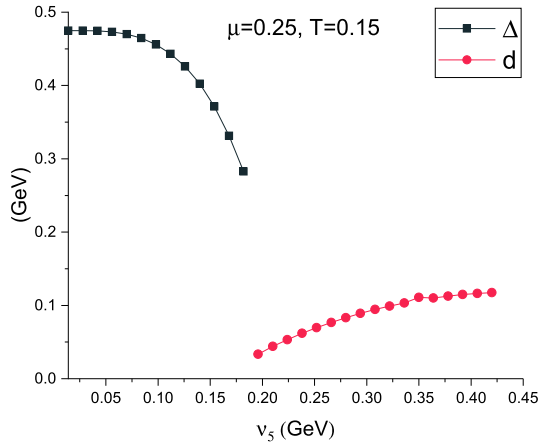


Fig. 3. (color online) Scalar diquark condensate Δ and axial vector diquark d as a function of v_5 for a given μ and T . The units of μ and T are GeV.

broken by the scalar diquark condensate, as presented in Fig. 3 for chemical potential $\mu = 250$ MeV. The scalar diquark condensate vanishes at high axial isospin density. Simultaneously, the condensation of the axial vector diquark begins. At low temperature $T = 40$ MeV, and if the chemical potential is lower than 150 MeV or higher than 300 MeV, the value of the chiral condensate, *i.e.* the sum of π_3 and Δ , is continuous and disappears for dense v_5 . In turn, the system is characterized by the axial vector diquark condensation of d , as is shown in Fig. 4.

5 Conclusions

In this paper, we proposed that an axial isospin density term $v_5 \bar{\psi} \gamma_0 \gamma_5 \tau_3 \psi$ is included in the Lagrangian [62–64] to mimic the parallel or anti-parallel EM fields in quark matter. The underlying mechanism is the following: because of their asymmetric electric charges, the u, d quarks induce an intense electromagnetic triangle anomaly if the EM field strength is at the hadron scale. As we are inter-

ested in the pattern of quark-quark pairing, we considered in this work a system at finite baryon chemical potential μ , starting from 70 MeV. In order to ease the comparison with lattice calculations, we have adopted a two-color QCD toy model to guarantee the positivity of the Euclidean path integral with pseudo-real fermions [4, 7, 8]. In this QCD-like theory, the NJL-type interactions are embedded in the extended global $SU(4)$ symmetry.

Following previous works on anisotropic dense quark matter [4, 7, 8], the chiral rotation in $SU(4)$ space is studied using the low energy effective theory χ PT at leading order. Using the covariance of the gauge symmetry, we obtained the possible QCD vacuum orientations in several environments. It was shown that there is a qualitative difference between dense baryon systems at finite isospin and axial isospin densities, which originates in the distinct nature of quasi-particles, mesons and baryons (diquarks). We analytically computed the alignment of the ground state to the scalar diquarks, $\Delta(\Delta^*)$, axial-vector diquarks, $d(d^*)$, and neutral pion meson directions from the usual scalar meson σ condensate in the $\mu - v_5$ plane. The possible states of the pion superfluid agree with the results obtained in [56], where the chiral rotation to π_3 is driven by the electromagnetic anomaly. We emphasize that the neutral π_3 and charged π_{\pm} lose their degeneracy in the EM fields. It was suggested that the system favors the charged pion instead of neutral pion condensation if more EM anomaly processes are included, such as the box anomaly or the five-point contribution $\gamma\gamma \rightarrow \pi_3\pi_+\pi_-$ [79, 80]. Therefore, it is reasonable that there is a discrepancy between Eq. (23) and the results of charged pion condensation obtained in [75]. In our case, the charged pion condensate is absent for nonzero v_5 since the axial isospin current is generated by a single triangle anomaly.

Assuming that the condensates involve only real Δ , d and π_3 , and applying the mean-field approximation, we studied the phase structure of the two-color NJL model. It

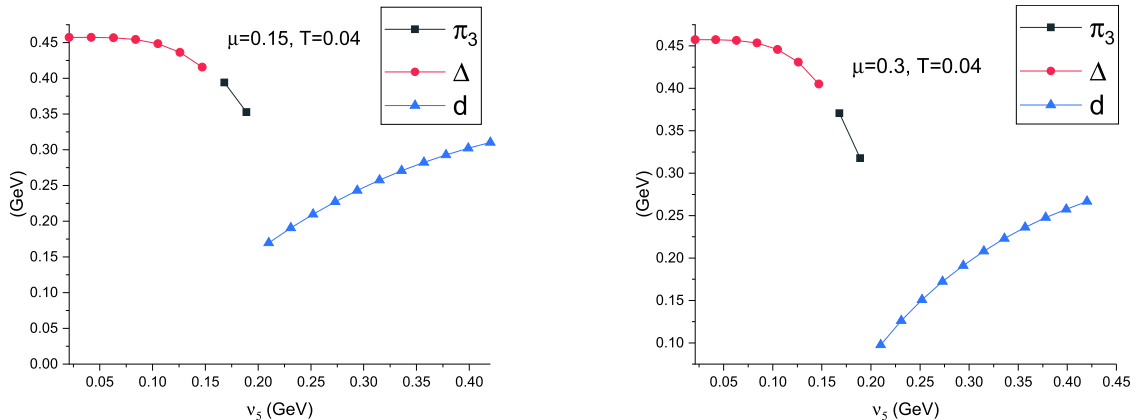


Fig. 4. (color online) Neutral pion condensate π_3 , scalar diquark condensate Δ and axial vector diquark d as a function of v_5 for a given μ and T . The units of μ and T are GeV.

was shown that the model calculations are consistent with the expectations from χ PT. We remark that a novel result was revealed in the QCD phase diagram, where a co-existent phase appears. It mixes the neutral pion and scalar diquark condensates in a shallow window where μ is proportional to v_5 . The relevant thermodynamic quantities at finite temperatures were obtained as well. Using either the χ PT or NJL models, the two-color calculations can be applied easily to the three-color QCD.

In our simplified model, the back reactions on the electric blind fields have not been taken into account. The coupling of the quark sector to the Polyakov loop is not included and therefore there is no (de)confinement effect [81-83]. Furthermore, a first principles calculation could be pursued by the Dyson-Schwinger equations and/or functional renormalization group approach. As this study was devoted to the properties of the axial isotopically asymmetric dense QCD matter, it could be verified by lattice simulations of the two-color QCD applying similar strategies for finite baryon and isospin chemical potentials.

An application of our analysis to heavy-ion collisions,

where strong anti-parallel electric and magnetic fields are produced above and below the reaction plane, is suggested. In addition, we observed that the axial vector diquark condensate develops as a physical ground state in the phase diagram. Its equation-of-state, thermodynamic behavior and transport coefficients, especially its emissivity, could be studied in low-temperature/high-density heavy-ion collision experiments at several worldwide facilities. It also gives reason for further investigations in the case of compact stars.

To summarize, this work explores the phenomenology of the QCD-like matter in different extreme conditions. The study of the two-color QCD serves as a first step towards realistic QCD matter. The subject of a future investigation will be to extend this work to the $N_c = 3$ QCD matter. We believe that our effort contributes to the general understanding of cold dense strongly interacting matter.

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References

- 1 M. G. Alford, A. Schmitt, K. Rajagopal et al, *Rev. Mod. Phys.*, **80**: 1455-1515 (2008)
- 2 T. D. Cohen, density, *Phys. Rev. Lett.*, **91**: 222001 (2003)
- 3 K. Splittorff and J. J. M. Verbaarschot, *Phys. Rev. D*, **75**: 116003 (2007)
- 4 M. Peskin, *Nuclear Physics B*, **175**(2): 197-233 (1980)
- 5 J. Verbaarschot, *Phys. Rev. Lett.*, **72**: 2531-2533 (1994)
- 6 S. Hands, J. B. Kogut, M. P. Lombardo et al, *Nuclear Physics B*, **558**(1): 327-346 (1999)
- 7 J. B. Kogut, M. A. Stephanov, D. Toublan et al, *Nuclear Physics B*, **582**(1): 477-513 (2000)
- 8 K. Splittorff, D. T. Son, and M. A. Stephanov, *Phys. Rev. D*, **64**: 016003 (2001)
- 9 G. V. Dunne and S. M. Nishigaki, *Nuclear Physics B*, **670**(1): 307-328 (2003)
- 10 S. Muroya, A. Nakamura, and Nonaka C, *Physics Letters B*, **551**(3): 305-310 (2003)
- 11 C. Ratti and W. Weise, *Phys. Rev. D*, **70**: 054013 (2004)
- 12 B. B. Alles, M. D'Elia, and M.P. Lombardo, *Nuclear Physics B*, **752**(1): 124-139 (2006)
- 13 S. Chandrasekharan and F. J. Jiang, *Phys. Rev. D*, **74**: 014506 (2006)
- 14 N. Strodthoff, B. J. Schaefer, and L. von Smekal, *Phys. Rev. D*, **85**: 074007 (2012)
- 15 V. V. Braguta, E.-M. Ilgenfritz, A.Yu. Kotov et al, *Phys. Rev. D*, **94**: 114510 (2016)
- 16 D. T. Son and M. A. Stephanov, *Phys. Rev. Lett.*, **86**: 592-595 (2001)
- 17 J. B. Kogut and D. K. Sinclair, *Phys. Rev. D*, **66**: 014508 (2002)
- 18 B. Klein, D. Toublan, and J. J. M. Verbaarschot, *Phys. Rev. D*, **68**: 014009 (2003)
- 19 B. B. Brandt, G. Endrődi, and S. Schmalzbauer, *Phys. Rev. D*, **97**: 054514 (2018)
- 20 B.-A. Li, L.-W. Chen, and C. M. Ko, *Physics Reports*, **464**: 113-281 (2008)
- 21 Z. Xiao, L.-W. Chen, F. Fu et al, *Journal of Physics G: Nuclear and Particle Physics*, **36**: 064040 (2009)
- 22 L. Engvik, M. Hjorth-Jensen, E. Osnes et al, *Phys. Rev. Lett.*, **73**: 2650 (1994)
- 23 D. Toublan and J. B. Kogut, *Physics Letters B*, **564**(3): 212-216 (2003)
- 24 A. Barducci, R. Casalbuoni, G. Pettini et al, *Phys. Rev. D*, **69**: 096004 (2004)
- 25 L. He, M. Jin, and P. Zhuang, *Phys. Rev. D*, **71**: 116001 (2005)
- 26 H. Abuki, R. Anglani, R. Gatto et al, *Phys. Rev. D*, **79**: 034032 (2009)
- 27 G. Cao, L. He, and X.-G. Huang, *Chinese Physics C*, **41**: 051001 (2017)
- 28 J. O. Andersen and T. Brauner, *Phys. Rev. D*, **78**: 014030 (2008)
- 29 H. Ueda, T. Z. Nakano, A. Ohnishi et al, *Phys. Rev. D*, **88**: 074006 (2013)
- 30 R. Stiele, E. S. Fraga, and J. Schaffner-Bielich, isospin, *Physics Letters B*, **729**: 72-78 (2014)
- 31 Z. Zhang and Y. X. Liu, *Phys. Rev. C*, **75**: 035201 (2007)
- 32 E. E. Svanes and J. O. Andersen, *Nuclear Physics A*, **857**(1): 16-28 (2011)
- 33 K. Kamikado, N. Strodthoff, L. von Smekal et al, *Physics Letters B*, **718**(3): 1044-1053 (2013)
- 34 S. P. Klevansky and R. H. Lemmer, *Phys. Rev. D*, **39**: 3478-3489 (1989)
- 35 A. Y. Babansky, E. V. Gorbar, and G. V. Shchepanyuk, *Physics Letters B*, **419**(1): 272-278 (1998)
- 36 E. J. Ferrer, V. de la Incera, and C. Manuel, *Phys. Rev. Lett.*, **95**: 152002 (2005)
- 37 J. L. Noronha and I. A. Shovkovy, *Phys. Rev. D*, **76**: 105030 (2007)
- 38 D. T. Son and M. A. Stephanov, *Phys. Rev. D*, **77**: 014021 (2008)
- 39 B. Feng, D. F. Hou, H. C. Ren et al, *Phys. Rev. Lett.*, **105**: 042001 (2010)
- 40 G. Cao, L. He, and P. Zhuang, *Phys. Rev. D*, **90**: 056005 (2014)
- 41 The STAR Collaboration, *Phys. Rev. Lett.*, **103**: 251601 (2009)
- 42 R. C. Duncan and C. Thompson, *The Astrophysical Journal*

- [Letters](#), **392**: L9-L13 (1992)
- 43 D. Grasso and H. R. Rubinstein, [Physics Reports](#), **348**(3): 163-266 (2001)
- 44 The ALICE Collaboration, [Phys. Rev. Lett.](#), **110**: 012301 (2013)
- 45 W. T. Deng and X. G. Huang, [Phys. Rev. C](#), **85**: 044907 (2012)
- 46 V. A. Miransky and I. A. Shovkovy, [Physics Reports](#), **576**: 1-209 (2015)
- 47 V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, [Phys. Rev. Lett.](#), **73**: 3499-3502 (1994)
- 48 G. S. Bali, F. Bruckmann, G. Endrodi et al, [Phys. Rev. D](#), **86**: 071502 (2012)
- 49 J. Chao, P. Chu, and M. Huang, [Phys. Rev. D](#), **88**: 054009 (2013)
- 50 F. Bruckmann, G. Endrődi, and T. Kovács, [Journal of High Energy Physics](#), **2013**(4): 112 (2013)
- 51 M. N. Chernodub, [Phys. Rev. D](#), **82**: 085011 (2010)
- 52 Y. Hidaka and A. Yamamoto, [Phys. Rev. D](#), **87**: 094502 (2013)
- 53 H. Liu, and L. Yu, and M. Huang, [Phys. Rev. D](#), **91**: 014017 (2015)
- 54 G. S. Bali, B. B. Brandt, G. Endrődi et al, [Phys. Rev. D](#), **97**: 034505 (2018)
- 55 G. Cao, [Phys. Rev. D](#), **100**: 074024 (2019)
- 56 G. Cao and X. G. Huang, [Phys. Lett. B](#), **757**: 1-5 (2016)
- 57 M. Ruggieri and G. X. Peng, [Phys. Rev. D](#), **93**: 094021 (2016)
- 58 L. Wang, and G. Cao, X.-G. Huang et al, [Phys. Lett. B](#), **780**: 273-282 (2018)
- 59 K. Fukushima, D. E. Kharzeev, and H. J. Warringa, [Phys. Rev. D](#), **78**: 074033 (2008)
- 60 D. E. Kharzeev and H. U. Yee, [Phys. Rev. D](#), **83**: 085007 (2011)
- 61 X. G. Huang and J. Liao, [Phys. Rev. Lett.](#), **110**: 232302 (2013)
- 62 S. L. Adler, [Phys. Rev.](#), **177**: 2426-2438 (1969)
- 63 J. S. Bell and R. Jackiw, [Il Nuovo Cimento A \(1965-1970\)](#), **60**(1): 47-61 (1969)
- 64 R. Kaiser, [Phys. Rev. D](#), **63**: 076010 (2001)
- 65 D. Ebert, T. G. Khunjua, and K. G. Klimenko, [Phys. Rev. D](#), **94**: 116016 (2016)
- 66 T. G. Khunjua, K. G. Klimenko, and R. N. Zhokhov, [Phys. Rev. D](#), **97**: 054036 (2018)
- 67 O. Janssen, M. Kieburg, K. Splitteroff et al, [Phys. Rev. D](#), **93**: 094502 (2016)
- 68 T. Schäfer, [Phys. Rev. D](#), **62**: 094007 (2000)
- 69 M. G. Alford, J. A. Bowers, J. M. Cheyne et al, [Phys. Rev. D](#), **67**: 054018 (2003)
- 70 M. Buballa, J. Hošek, and M. Oertel, [Phys. Rev. Lett.](#), **90**: 182002 (2003)
- 71 A. Schmitt, Q. Wang, and D. H. Rischke, [Phys. Rev. Lett.](#), **91**: 242301 (2003)
- 72 D. N. Aguilera, D. Blaschke, M. Buballa et al, [Phys. Rev. D](#), **72**: 034008 (2005)
- 73 F. Marhauser, D. Nickel, M. Buballa et al, [Phys. Rev. D](#), **75**: 054022 (2007)
- 74 X. Wang and I. A. Shovkovy, [Phys. Rev. D](#), **82**: 085007 (2010)
- 75 J. Chao, M. Huang, and A. Radzhabov. arXiv: 1805.00614, 2018
- 76 T. A. Rytov and F. Sannino, [Phys. Rev. D](#), **78**: 115010 (2008)
- 77 G. Cacciapaglia and F. Sannino, [Journal of High Energy Physics](#), **2014**(4): 111 (2014)
- 78 M. Huang and I. Shovkovy, [Nuclear Physics A](#), **729**(2): 835-863 (2003)
- 79 S. L. Adler, B. W. Lee, S. B. Treiman et al, [Phys. Rev. D](#), **4**: 3497-3501 (1971)
- 80 M. V. Terent'ev, [Soviet Journal of Experimental and Theoretical Physics Letters](#), **14**: 94 (1971)
- 81 A. Amador and J. O. Andersen, [Phys. Rev. D](#), **88**: 025016 (2013)
- 82 D. Blaschke, A. Dubinin, and M. Buballa, [Phys. Rev. D](#), **91**: 125040 (2015)
- 83 V. G. Bornyakov, V. V. Braguta, E. -M. Ilgenfritz et al, [Journal of High Energy Physics](#), **2018**(3): 161 (2018)