H→γγ: a comment on the indeterminacy of non-gauge-invariant integrals

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Abstract: We reanalyze the recent computation of the amplitude of the Higgs boson decay into two photons presented by Gastmans et al. [1, 2]. The reasons for why this result cannot be the correct one have been discussed in some recent papers. We address here the general issue of the indeterminacy of integrals with four-dimensional gauge-breaking regulators and to which extent it might eventually be solved by imposing physical constraints. Imposing gauge invariance as the last step upon R_ξ-gauge calculations with four-dimensional gauge-breaking regulators, allows us to recover the well known $\mathrm{H}\to\gamma\gamma$ result. However we show that in the particular case of the unitary gauge, the indeterminacy cannot be tackled in the same way. The combination of the unitary gauge with a cutoff regularization scheme turns out to be non-predictive.

Key words: Higgs decay, regularization methods, loop calculations, gauge invariance

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1 Introduction

Recently some attention has been re-focused on the W-loop contribution in the calculation of the $\mathrm{H}\to\gamma\gamma$ amplitude because of a result presented by Gastmans et al. [1, 2] turning out to be at odds with the renowned one of Refs. [3, 4]. It goes without saying that, if correct, the result in Refs. [1, 2] would have had relevant consequences for the ongoing Higgs boson searches at the LHC.

Starting from the observation that the full amplitude $\mathrm{H}\to\gamma\gamma$ is free of ultraviolet and infrared singularities, Gastmans et al. performed their calculation in four dimensions with no regulators and used the unitary gauge to consider only the physical degrees of freedom. A gauge invariant amplitude is obtained with the 'Dyson subtraction' [5], leading to

$$\mathcal{M} = \frac{e^2 g}{(4\pi)^2 m_\mathrm{W}}[3\tau+3\tau(2-\tau)f(\tau)] \times(k_1\cdot k_2 g^{\mu\nu}-k_2^\mu k_1^\nu)\epsilon_\mu(k_1)\epsilon_\nu(k_2), \quad (1)$$

where $\tau=\dfrac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}$ and

$$f(\tau)=\begin{cases}\arcsin^2(\tau^{-\frac{1}{2}}) & \text{for} \quad \tau\geqslant 1\\ -\dfrac{1}{4}\left[\ln\dfrac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}}-\mathrm{i}\pi\right]^2 & \text{for} \quad \tau<1\end{cases}. \quad (2)$$

This amplitude, which happens to vanish in the $m_\mathrm{W}/m_\mathrm{H}\to 0$ limit (contrary to the standard one), would imply a reduction of the decay width $\Gamma(\mathrm{H}\to\gamma\gamma)$ by about 50% for $m_\mathrm{H}\approx 120$ GeV, with respect to that found in Refs. [3, 4].

The standard $\mathrm{H}\to\gamma\gamma$ amplitude was computed in 't Hooft-Feynman gauge with dimensional regularization [3], background field methods [4], and in the unitary gauge with renormalization group analysis [4]. It reads as

$$\mathcal{M} = \frac{e^2 g}{(4\pi)^2 m_\mathrm{W}}[2+3\tau+3\tau(2-\tau)f(\tau)] \times(k_1\cdot k_2 g^{\mu\nu}-k_2^\mu k_1^\nu)\epsilon_\mu(k_1)\epsilon_\nu(k_2). \quad (3)$$

Gastmans et al. cast some doubt on the reliability of using dimensional regularization and prefer their result with the motivation that it would respect some $m_\mathrm{W}/m_\mathrm{H}\to 0$ 'decoupling limit', which has indeed no reason to hold true, as explained in Refs. [6–9] in the framework of the equivalence theorem [10].

The results of Refs. [1, 2] have been criticized by a number of recent papers [6–9, 11–13].

The criticism concerns the absence of regulators, leading to ambiguities in the intermediate steps of the calculation, the use of Dyson subtraction and the reference to the Appelquist-Carazzone theorem [14] to justify the decoupling.

The amplitude $H\to\gamma\gamma$ has been calculated in several ways, all confirming the old result of Refs. [3, 4], as

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follows:

1) The authors of Ref. [7] redid the calculation in four dimensions in the unitary gauge with a gauge-invariant regularization method (Pauli-Villars like) [15]. They stress the importance of having set an explicit regularization scheme to control finite terms which guarantee gauge invariance through all intermediate steps of the calculation. The authors cross-check the calculation with an independent one in dimensional regularization and underscore that no renormalization condition should be applied in the presence of only finite terms. Thus they conclude that the calculation presented by Gastmans et al. must be wrong because it is finite and not gauge-invariant.

2) The authors of Ref. [8] perform the calculation in dimensional regularization both in the unitary and in R_ξ gauges (the same as in Ref. [9]). It is pointed out that, without any regulator, the coefficient of $g^{\mu\nu}$, arising upon four-dimensional symmetric integration in renormalizable gauges, is an indeterminate form of the kind $\infty-\infty$ – also responsible of the breaking of gauge invariance – and that in the unitary gauge the same happens to the coefficient of $k_2^\mu k_1^\nu$.

3) H. S. Shao et al. [11] perform the calculation using a four-dimensional momentum cutoff regularization both in the ‘t Hooft-Feynman and unitary gauges. Within the latter they obtain the same result of Refs. [1, 2] starting with a particular routing of momenta. They also get the terms to be added after a shift in the loop momentum: these contributions sum up to zero when the momentum choice of Refs. [1, 2] is adopted. Performing the calculation in the ‘t Hooft-Feynman gauge, the authors recover the same gauge invariant result as the one obtained in dimensional regularization [3] by subtracting the contribution of all non-constant diagrams (evaluated at $k_1=k_2=0$). The result is independent of the loop momentum choice because the divergences are only logarithmic.

4) F. Bursa et al. [13] perform the calculation of the $\mathrm{H}\to\gamma\gamma$ decay amplitude using a (gauge invariant) space-time lattice regulator and obtain a very good numerical agreement with the decay amplitude evaluated with dimensional regularization.

Summarizing, in Ref. [8] it is highlighted that the problem of Refs. [1, 2] resides in the absence of regulators. In Ref. [11], however, it is shown that the use of cutoff regularization in the unitary gauge leads to confirm the result by Gastmans et al. Thus we might observe that, if there is a problem in the latter calculation, it is not in the lack of a regulator but rather in the combination of the unitary gauge with the use of non-gauge invariant regulators. As the cutoff regularization has been widely used in the literature, we explore further its connection with the unitary gauge: in particular we attempt to get a deeper understanding of the result presented in Ref. [11].

2 The critical integrals in the $\mathbf{H}\to\gamma\gamma$ amplitude

At the core of the problem of the $\mathrm{H}\to\gamma\gamma$ amplitude calculation is the calculation of the integral

$$I_{\mu\nu}=\int \mathrm{d}^4 l\,\frac{g_{\mu\nu}l^2-4l_\mu l_\nu}{\left(l^2-M^2+\mathrm{i}\epsilon\right)^3}, \tag{4}$$

where $M^2=m_{\mathrm{W}}^2-x_1x_2m_{\mathrm{H}}^2$, and x_1,x_2 are Feynman parameters. According to Gastmans et al., performing the integral in four dimensions with symmetric boundaries[1)], we can substitute $l_\mu l_\nu\to\frac{1}{4}l^2 g_{\mu\nu}$, leading to $I_{\mu\nu}=0$. Here we can appreciate the difference with respect to dimensional regularization (DR), where

$$I^{\mathrm{DR}}_{\mu\nu}(n)=\int \mathrm{d}^n l\,\frac{g_{\mu\nu}l^2-4l_\mu l_\nu}{\left(l^2-M^2+\mathrm{i}\epsilon\right)^3}=-\mathrm{i}g_{\mu\nu}\frac{\pi^2}{2}+O(n-4). \tag{5}$$

Gastmans et al. conclude that $I^{\mathrm{DR}}_{\mu\nu}(n)$ must have a discontinuity in $n=4$, thus mining the foundations of the DR technique stating that integrals are not analitic in n dimensions.

Let us start from the four-dimensional integral in Eq. (4). After Wick rotation[2)] and rescaling $l\to l/M$, we get

$$I_{\mu\nu}=\mathrm{i}\int \mathrm{d}^4 l\,\frac{\delta_{\mu\nu}l^2-4l_\mu l_\nu}{\left(l^2+1\right)^3}. \tag{6}$$

To simplify the discussion, we focus on the case $\mu=1$, $\nu=1$

$$I_{11}=\mathrm{i}\int \mathrm{d}^4 l\,\frac{l^2-4l_1^2}{\left(l^2+1\right)^3}=\mathrm{i}\int \mathrm{d}^4 l\,F_{11}(l). \tag{7}$$

The integrand is not a summable function: it is not positive everywhere in the domain of integration, and $\int \mathrm{d}^4 l\,|F_{11}|=\infty$, which means that the integral is not defined per se - the value depends on how the boundary is chosen to behave at infinity. We therefore compute the value of the integral over different integration domains with different behaviors at infinity: we will observe how the integral may assume every finite value, and even diverge.

As a first example, let us consider a ‘spherical cutoff’ in the sense described below. In polar coordinates, we

1) Since the integral in Eq. (4) does not depend on any external momentum, for tensor invariance it must be $I_{\mu\nu}=Ig_{\mu\nu}$; by saturating both sides with $g^{\mu\nu}$, we have $I_{\mu\nu}g^{\mu\nu}\to 4I$, $l^2g_{\mu\nu}\to 4l^2$ and $l_\mu l_\nu\to l^2$, then we can solve with respect to I. We have the same result if we substitute $l_\mu l_\nu\to\frac{1}{4}l^2$.

2) Which amounts to the following substitutions: $\mathrm{d}^4l\to\mathrm{i}\mathrm{d}^4l$, $g_{\mu\nu}\to-\delta_{\mu\nu}$, $l^2\to l_E^2$ and $l_\mu\to l_\mu^E$.

write

$$I_{11} = \mathrm{i}\int_0^{\Lambda}\mathrm{d}l\,\frac{l^5}{(1+l^2)^3}\int\mathrm{d}\Omega_4\,(1-4\cos^2\theta)$$
$$= \mathrm{i}4\pi\int_0^{\Lambda}\mathrm{d}l\,\frac{l^5}{(1+l^2)^3}\int_0^{\pi}\mathrm{d}\theta\sin^2\theta(1-4\cos^2\theta)=0. \quad (8)$$

Λ is an adimensional cutoff, with l being an adimensional integration variable. The angular part vanishes, so there are no problems with the logarithmic divergence of the radial part. Actually, every integration domain which has the $l_i \leftrightarrow \pm l_j$ symmetry, leads to an identically vanishing integral.

As a second case we choose a non-symmetrical domain of integration. For example, we integrate F_{11} over the elliptical domain $\frac{l_1^2}{1+\epsilon}+l_2^2+l_3^2+l_4^2 \leqslant \Lambda^2$ (see Fig. 1)

$$I_{11} = \mathrm{i}4\pi\sqrt{1+\epsilon}\int_0^{\Lambda}\mathrm{d}l\,l^5\int\mathrm{d}\theta\sin^2\theta \times\frac{1-(4+3\epsilon)\cos^2\theta}{(1+l^2+l^2\epsilon\cos^2\theta)^3}\xrightarrow{\Lambda\to\infty}\mathrm{i}\pi^2\frac{8+4\epsilon-\epsilon^2-8\sqrt{1+\epsilon}}{2\epsilon^2}. \quad (9)$$

The integral in Eq. (9) can assume different finite values as a function of ϵ.

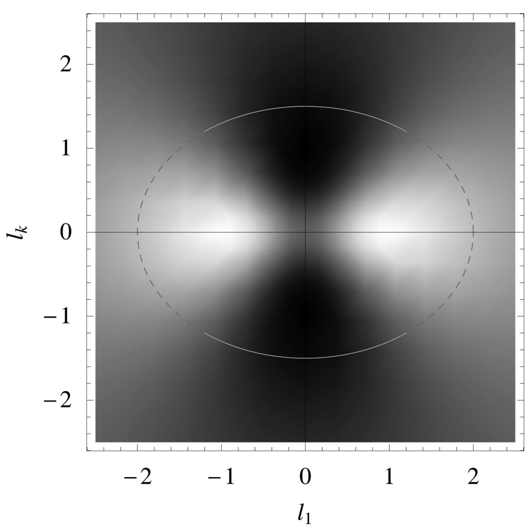

Fig. 1. We integrate $F_{11}=(l^2-4l_1^2)/(1+l^2)^3$ over an elliptic domain. Because of the cylindrical symmetry, the graphic is the same independent of $l_k = l_2, l_3, l_4$. The darker the background is, the larger the F_{11} value. The boundary is solid when $F_{11}>0$, and is otherwise dashed.

Choosing asymmetric boundaries, we even loose tensor invariance, obtaining a 4×4 matrix of unrelated, indeterminate terms. This translates into the fact that $I_{\mu\nu}$ is no longer proportional to $\delta_{\mu\nu}$ as it should be the case (see Eq. (6)). We seek an appropriate choice of the boundaries for all the terms in the $I_{\mu\nu}$ matrix in such a way to recover a $\delta_{\mu\nu}$ structure. We can therefore compute the $I_{\mu\nu}$ entries by choosing the same asymmetric boundary on all diagonal terms, and generic symmetric boundaries for all off-diagonal terms. In this way, all diagonal terms will have the same indeterminate value I, whereas off-diagonal terms will vanish. We thus obtain $I_{\mu\nu}=I\delta_{\mu\nu}$, with I being an indeterminate (even divergent) constant. In Appendix A we give more details on this. We also consider the case of Schwinger regularization to remark that the indeterminacy of the critical integrals in these calculations can be solved only with physical constraints as there is no mathematical prescription which can univocally determine them.

Gastmans et al. rely on their finite (equal to zero) result for the integral of the type of Eq. (4), which follows from a particular choice of the integration domain. This also explains why the calculation of Ref. [11], in which a spherical cutoff is explicitly used, leads to the same result found by Gastmans et al. in unitary gauge: the choice of the integration domain in Ref. [11] is the same as the one implicitly taken in Refs. [1, 2].

The authors of Ref. [8], on the other hand, underscore the fact that the integral (4) is an indeterminate form of the kind ∞–∞: it must be treated with some regularization scheme. Gauge-invariance in the final result can be implemented either a priori by choosing gauge-invariant regulators (like Pauli-Villars or DR), or a posteriori by applying an appropriate subtraction. We remark that in the latter case integrals are not well defined, and their values must be considered indeterminate.

If we choose a sharp spherical cutoff we still get the Gastmans et al. result in a unitary gauge as was first shown in Ref. [11]. On the other hand, if we use renormalizable gauges in a cutoff scheme, we recover the standard result (a ‘t Hooft-Feynman gauge is used in Ref. [11]).

Does the fact that two different results are obtained using two different gauge choices mean that a cutoff scheme is not to be pursued at all? In the following we show that the problem does not reside in the use of a cutoff scheme by itself but rather in figuring out that different ways (spherical, elliptical, etc.) of implementing a cutoff scheme amount to different values of the integrals, i.e., to indeterminate coefficients. We can actually use some cutoff schemes provided that there is a clear recipe on how to absorb the indeterminate coefficients arising in the calculation, also restoring gauge invariance. We will show that such a recipe cannot be found in the unitary gauge.

3 Considerations on power counting and gauge invariance

Jackiw [16] showed that indeterminacy can arise if we use regulators which have less symmetry than the theory, and it is not necessarily resolved when we restore the symmetry at the end of the calculation. We can simply understand the case of gauge symmetry: gauge invariant regulators decrease the degree of divergence of the integral, making it finite and regulator independent. Indeed, by naive power counting, we know that the amplitude $\mathrm{H}\to\gamma\gamma$ is logarithmically divergent in renormalizable gauges. In gauge invariant regularizations, we can group two momentum powers in the numerator to extract the gauge invariant factor $k_1{\cdot}k_2 g^{\mu\nu}-k_2^{\mu}k_1^{\nu}$, so that the amplitude becomes finite. Also in the unitary gauge we expect the same finite amplitude after a not-straightforward cancellation of higher divergent terms. However, this cannot be done in cutoff regularization where the Ward identity and gauge invariance are spoiled by the breaking of shift invariance. The expectations of Gastmans et al. to get a finite amplitude which needs no regulator are disappointed by the choice of four-dimensional symmetric integration, which implicitly uses a spherical cutoff scheme, leading to the breaking of gauge invariance and to a divergent amplitude.

To subtract the divergence in the cutoff regularization scheme and, in general, all cutoff-dependent terms, we need some counterterms. Breaking gauge symmetry, we have the most general lagrangian with all possible combinations of bare fields and bare couplings, even an ad hoc counterterm of the form $\delta m_{0A}h_0A_0^2$. This is what Dyson subtraction means: hide all divergent, cutoff-dependent, non-gauge-invariant terms into a counterterm, which would vanish in a gauge invariant regularization scheme.

We have computed the $\mathrm{H}\to\gamma\gamma$ amplitude in the 't Hooft-Feynman gauge without calculating divergent integrals, we find:

$$\begin{aligned}\mathcal{M}^{\mu\nu}_{\xi=1} = \frac{e^2g}{(4\pi)^2m_\mathrm{W}}\Bigg[&-k_2^{\mu}k_1^{\nu}\left(2+3\tau+3\tau(2-\tau)f(\tau)\right)\\ &-2m_\mathrm{H}^2\left(1+\frac{3}{2}\tau\right)\int_0^1\mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\int\frac{\mathrm{d}^4l}{\mathrm{i}\pi^2}\\ &\times\frac{g^{\mu\nu}l^2-4l^{\mu}l^{\nu}}{(l^2-1+4x_1x_2\tau+\mathrm{i}\epsilon)^3}\\ &+\frac{1}{2}m_\mathrm{H}^2g^{\mu\nu}\left(1+\frac{3}{2}\tau+3\tau(2-\tau)f(\tau)\right)\Bigg]. \qquad (10)\end{aligned}$$

We remark that the first term (proportional to $k_2^{\mu}k_1^{\nu}$) contains only well-defined finite integrals; the second term is indeterminate (vanishing according to symmetric integration as in Refs. [1, 2]). With the use of DR, the second term would give $\frac{1}{2}m_\mathrm{H}^2g^{\mu\nu}\left(1+\frac{3}{2}\tau\right)$, leading to the well known gauge-invariant expression. However, let us stay in the framework of gauge-breaking regularizations.

In Ref. [11] a modified version of Dyson subtraction is performed to recover gauge invariance. One might even wonder whether Dyson subtraction is allowed without divergent terms [7]. As we have just shown, the integral in the second term is probably divergent and in any case cutoff-dependent, so we are allowed to add a counterterm and impose gauge invariance as a renormalization condition. In doing so we get the correct amplitude in Eq. (3). We would have the same expression by using symmetric integration: every value of the integral disappears into the counterterm. We therefore conclude that the arbitrariness related to the choice of the boundary (or, in general, of the regulator) is solved by imposing gauge invariance.

Why is Gastmans et al. 's amplitude different from the standard one? As shown in Ref. [2], in a unitary gauge we have another divergent integral

$$\begin{aligned}A'=2\int_\mathrm{simplex}\mathrm{d}x_1\mathrm{d}x_2\int\mathrm{d}^4l\,\Big[&(k_2^{\mu}k_1^{\nu}-k_1{\cdot}k_2g^{\mu\nu})l^2\\ &-2k_1^{\nu}(k_2{\cdot}l)l^{\mu}-2k_2^{\mu}(k_1{\cdot}l)l^{\nu}+2(k_1{\cdot}k_2)l^{\mu}l^{\nu}\\ &+2g^{\mu\nu}(k_1{\cdot}l)(k_2{\cdot}l)\Big]\frac{1}{(l^2-m_\mathrm{W}^2+x_1x_2m_\mathrm{H}^2+\mathrm{i}\epsilon)^3} \qquad (11)\end{aligned}$$

By symmetric integration the integral vanishes, whereas dimensional regularization leads to $A'_\mathrm{DR}(n)=\mathrm{i}\pi^2(k_2^{\mu}k_1^{\nu}-k_1{\cdot}k_2g^{\mu\nu})+O(n-4)$. The integral has the same indeterminate behavior of the former one: the value depends on the choice of the boundary. We can say that $A'=Jk_2^{\mu}k_1^{\nu}+J'g^{\mu\nu}$, with J and J' indeterminate constants. While in Eq. (10) the tensor $k_2^{\mu}k_1^{\nu}$ has a well-defined finite coefficient and we can tune the rest of the amplitude on it, in unitary gauge this coefficient is indeterminate, possibly divergent: we must then add another counterterm $\delta g_{0A}h_0(\partial^{\mu}A_0^{\nu})^2$ to absorb the divergence.

We now have two counterterms and we need two renormalization conditions to fix the arbitrariness. The only Dyson subtraction (which means imposing gauge invariance) is not enough anymore. The result in Eq. (1) is still arbitrary, and allows the addition of whatever gauge invariant $k_2^{\mu}k_1^{\nu}-k_1{\cdot}k_2g^{\mu\nu}$ term. The other condition could be, for example, the requirement of the validity of the equivalence theorem [10] in the limit $m_\mathrm{W}\to0$, or the invariance of the amplitude in both the 't Hooft-Feynman and the unitary gauge: both conditions fix the value of the amplitude in Eq. (1) to the standard result in Eq. (3). The indeterminacy is solved; we recover also the independence of the amplitude on gauge choice and regulator choice.

4 Conclusions

We have analyzed the computation of the amplitude $H\to\gamma\gamma$ by Gastmans et al. [1, 2], to understand why it turns out to be different from the standard result in Eq. (3). Integrals of the form

$$I_{\mu\nu}=\int d^4l\,\frac{g_{\mu\nu}l^2-4l_\mu l_\nu}{(l^2-M^2+i\epsilon)^3}, \tag{12}$$

are not well defined. We have provided some explicit examples within cutoff regularization, obtaining different values by varying integration boundaries.

In the 't Hooft-Feynman gauge and in a cutoff regularization scheme, see Eq. (10), we obtain

$$\mathcal{M}^{\mu\nu}_{\xi=1}=\frac{e^2g}{(4\pi)^2m_W}\left[-k_2^\mu k_1^\nu\left(2+3\tau+3\tau(2-\tau)f(\tau)\right)+Ig^{\mu\nu}\right], \tag{13}$$

where I is a constant which depends on the boundary shape. This makes it indeterminate as there is no physical prescription on the choice of the integration boundary shape. On the other hand, the use of a gauge invariant regularization scheme automatically provides the recipe on how to evaluate the integrals.

Since the term $k_2^\mu k_1^\nu$, in Eq. (12), has only one finite unambiguous coefficient, we are able to solve the indeterminacy by imposing gauge invariance at the end of the calculation.

However, we have shown that, in the unitary gauge, both the coefficients of $k_2^\mu k_1^\nu$ and $g^{\mu\nu}$ are indeterminate in the sense of integration boundary shape dependency. Imposing only one renormalization condition (like imposing gauge invariance by Dyson subtraction) is not enough anymore. Given the equivalence of R_ξ gauges with a unitary gauge as $\xi\to\infty$, the problem we discuss is likely to be related to the exchange of this limit with an integral sign for non-Riemann-summable functions: the coefficient of $k_2^\mu k_1^\nu$ arises from highly divergent terms which do not appear at finite values of ξ.

Gastmans et al.'s expression in Eq. (1) is still ambiguous upon Dyson subtraction, and allows the addition of whatever term of the form $k_2^\mu k_1^\nu-k_1\cdot k_2 g^{\mu\nu}$. This arbitrariness can be fixed by requiring the validity of the equivalence theorem, or by imposing the equality of amplitudes in the unitary and 't Hooft-Feynman gauges. In other words, we are able to add terms to Eq. (1) in order to match the standard result Eq. (3).

The combination of unitary gauge with a cutoff regularization scheme simply turns out to be non-predictive.

Appendix A

For the sake of simplicity in the following we consider a two-dimensional version of $I_{\mu\nu}$ in Eq. (6)

$$I_{\mu\nu}=i\int d^2l\,\frac{\delta_{\mu\nu}l^2-2l_\mu l_\nu}{(l^2+1)^2}. \tag{A1}$$

This version of $I_{\mu\nu}$ has the same properties as its four-dimensional counterpart, namely: i) the integral is superficially divergent as a logarithm, ii) it is identically zero for symmetric integration domains, iii) the integrand function has no definite sign. The conclusions we will draw from the following calculations in two dimensions remain unaltered in four dimensions: here we have simply avoided superfluous technical complications.

As we did before, let us start by computing the I_{11} term. We realize that I_{11} can be mapped into an entry of the I_{12} kind upon a rotation by 45° of l_1l_2 axes. $I_{11}=I_{12}$ only if the integration domain is rotated accordingly. Since the calculations turn out to be simpler using the {12} entry, we will make our observations on this case only

$$I_{12}=i\int d^2l\,\frac{-2l_1l_2}{(1+l^2)^2}=i\int d^2l\,F_{12}. \tag{A2}$$

At any rate we remark that the domains of integrations will be chosen in such a way that eventually all off-diagonal $I_{\mu\nu}$ entries will vanish as to eventually recover the $\delta_{\mu\nu}$ tensor structure.

The integrand in Eq. (A2) is negative when $l_1l_2>0$ (I and III quadrant), and positive otherwise. In the former case, we bound a domain with two quarters of a circumference of radius Λ; in the latter case we use a square with edge Λ (Fig. A1 (left panel)). We have

$$\begin{aligned}I_{12} &= -4i\int_0^\Lambda dl\,\frac{l^3}{(1+l^2)^2}\int_0^{\pi/2}d\theta\sin\theta\cos\theta \\ &\quad +4i\int_{[0,\Lambda]\times[0,\Lambda]}dl_1dl_2\,\frac{l_1l_2}{(1+l_1^2+l_2^2)^2} \\ &= i\left(\frac{\Lambda^2}{1+\Lambda^2}+\ln\frac{1+\Lambda^2}{1+2\Lambda^2}\right)\xrightarrow{\Lambda\to\infty}i\left(1+\ln\frac{1}{2}\right).\end{aligned} \tag{A3}$$

Again we get a finite non-zero value. The leading divergences are the same in each quadrant, whereas the finite part is boundary-dependent, so that the sum does not vanish.

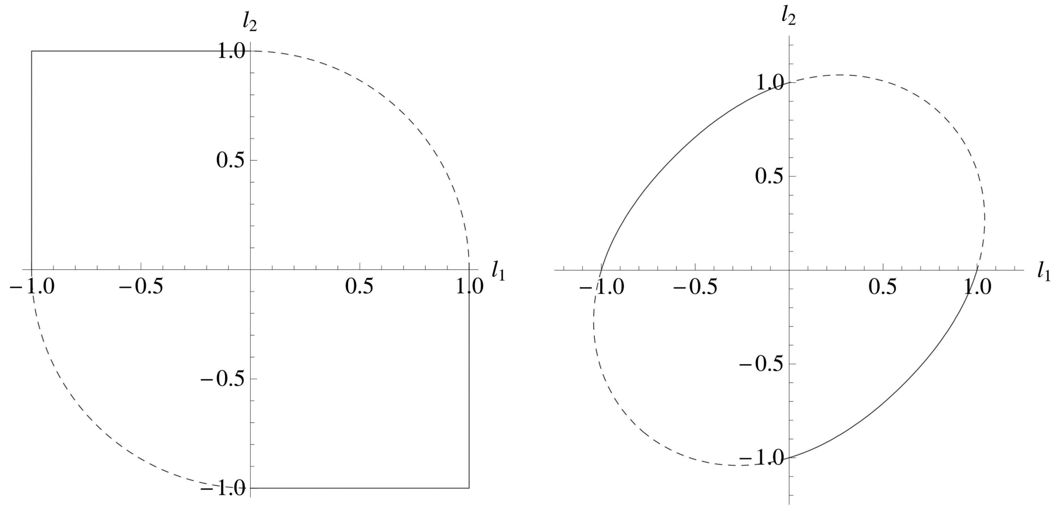

Fig. A1. We integrate $F_{12}=-2l_1l_2/\left(1+l^2\right)^2$ over a mixed (circle-square) boundary (left panel), regulating the function with a smooth cutoff Λ (right panel).

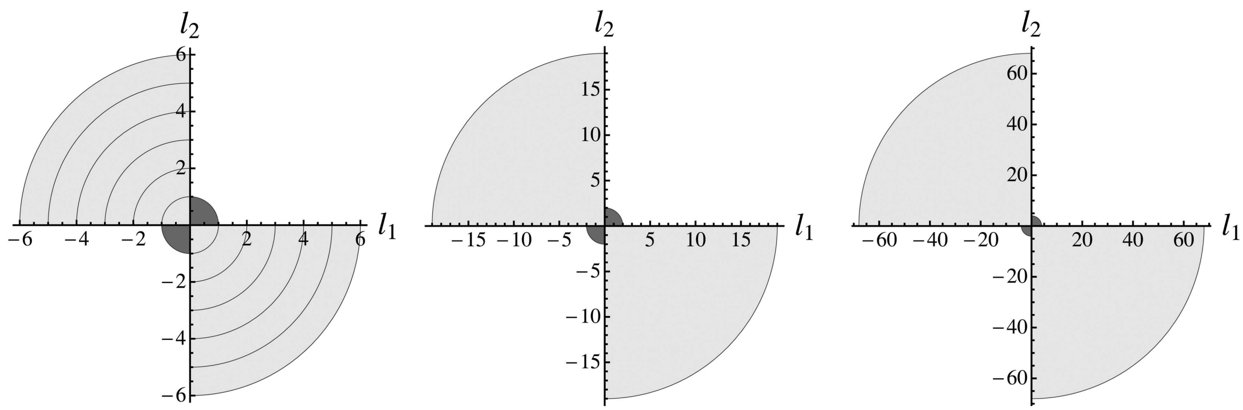

Fig. A2. Riemann rearrangement. Light gray regions have a positive integral p_k, dark gray regions have negative integrals n_k. We see that the negative region becomes smaller and smaller than the positive region, so that it cannot cancel the logarithmic divergence.

More in general, we can slice $\mathbb{R}^2$ in a countable set of bounded regions, in order to reduce the integral over the whole $\mathbb{R}^2$ to a countable sum of finite integrals, i.e., to a series. We can thus use the Riemann rearrangement theorem [17] to obtain whatever finite value or logarithmic divergence.

For example, let us consider all the concentric circumferences with integer radius, thus slicing $\mathbb{R}^2$ into annuli: the integral of F_{12} over each annulus vanishes by circular symmetry. Therefore we slice each annulus into a positive region P_k where $F_{12}>0$, and a negative region N_k where $F_{12}<0$ (Fig. A2). We therefore have

$$p_k=\int_{P_k}\mathrm{d}^2 l F_{12}=\frac{1}{k^2+2k+2}-\frac{1}{k^2+1}+\log\frac{k^2+2k+2}{k^2+1},$$

$$n_k=\int_{N_k}\mathrm{d}^2 l F_{12}=-p_k. \tag{A4}$$

The p_k form a bounded sequence of positive terms converging to 0. We can find that the greatest term of the sequence is $p_1=M\approx 0.62$. Specularly, the n_k form a sequence of negative terms converging to zero, bounded by $n_1=-M\approx-0.62$. If we take the union of all P_k and N_k, we recover the whole $\mathbb{R}^2$, therefore if we sum all p_k and n_k we recover the whole integral. Since both $\sum_k p_k$ and $\sum_k n_k$ diverge separately, we must specify the correct ordering of terms. We start by adding the first positive terms p_k until we exceed $1+M$, and then add the first negative term n_0. Since all n_k satisfy the relationship $-M<n_k<0$, we still have

$$p_0+p_1+\cdots+p_{N_1}-|n_0|>1. \tag{A5}$$

We can continue adding positive terms until we exceed $2+M$, and then add n_1, and so on. The resulting sum covers all P_k and N_k regions. The series diverges, and so does the integral.

One might wonder whether we have the same behavior with a smooth cutoff. We calculate Eq. (A2) with a Schwinger regulator [18]

$$I_{12} = \mathrm{i}\int \mathrm{d}^2 l(-2l_1 l_2)\int_{\frac{1}{\Lambda^2}}^{\infty} \mathrm{d}s s \mathrm{e}^{-s\left(1+l^2\right)}$$

$$= -\mathrm{i}\Gamma\left(0,\frac{1}{\Lambda^2}\right)\int_0^{2\pi} \mathrm{d}\theta \sin\theta\cos\theta=0, \qquad \text{(A6)}$$

where $\Gamma(a,b)$ is the incomplete Gamma function. Again, the angular part of the integral vanishes, and so we do not care about the logarithmic divergence in the radial part. However if we deform the cutoff giving an angular dependency to it, e.g. $\Lambda\rightarrow\Lambda(\theta)=\Lambda\exp(\epsilon\sin 2\theta)$ (Fig. A1 (right panel)), we obtain

$$I_{12} = \mathrm{i}\int \mathrm{d}^2 l(-2l_1 l_2)\int_{\frac{1}{\Lambda^2(\theta)}}^{\infty} \mathrm{d}s s \mathrm{e}^{-s\left(1+l^2\right)}$$

$$= -\mathrm{i}\int_0^{2\pi} \mathrm{d}\theta \sin\theta\cos\theta\,\Gamma\left(0,\frac{1}{\Lambda^2(\theta)}\right) \xrightarrow{\Lambda\rightarrow\infty} -\mathrm{i}\pi\epsilon. \qquad \text{(A7)}$$

Also in this case, the value of the integral depends on the shape of the cutoff function, no matter if smooth or sharp.

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