Another representation of the β form of the inhomogeneous Picard-Fuchs equation*

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Abstract: In this letter we give another representation of the β form in the inhomogeneous Picard-Fuchs equation for open topological string for some one-parameter Calabi-Yau hypersurfaces in weighted projective spaces. Furthermore, the corresponding domain wall tensions calculated by using these β forms are consistent with the results that appear in literature. The β form is essential for the calculation of the D-brane domain wall tension, and a convenient choice of β forms should simplify the calculation. The freedom of the choice of β forms shows some symmetries in Calabi-Yau space.

Key words: topologcial string theory, Calabi-Yau manifold, domain wall tension, D-brane

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1 Introduction

Topological string theory, which has mirror symmetry between its two different types–A model and B model, provides a powerful method for calculation of physical and mathematical data [1, 2]. From the viewpoint of mathematics, it can count curves on Calabi-Yau threefolds in closed topological string, which is Gromov-Witten invariants. While in the open topological string, which is linked to the D-brane, it can count holomorphic disks [3–5]. It can also relate to the Abel-Jacobi map, which serves as the domain wall tension of D-brane on the B model [3, 5, 6].

From the viewpoint of physics, it has applications in various aspects of superstrings and supersymmetric gauge theories. In the effective theory describing the Type II string Calabi-Yau compactifications, topological string theory could also be an important method. For instance, in closed topological string, it can be used for computing holomorphic prepotential. While in the open case, the superpotential related to D-brane can also be obtained in this way.

The domain wall tension $\mathcal{T}$ measures the difference of superpotential $\mathcal{W}$ for two D-brane configurations [7]. The key to solving the superpotential $\mathcal{W}$ is through solving the inhomogeneous Picard-Fuchs equations [5, 8–11]. Let X_z be a family of Calabi-Yau threefolds parameterized by one variable z, Ω_z and $\beta(z)$ be a family of non-zero holomorphic 3-forms and 2-form on X_z respectively. Let L be the Picard-Fuchs operator. Then the Picard-Fuch equation is

$$L\int_\Gamma \Omega(z)=-\int_\Gamma \mathrm{d}\beta(z). \tag{1}$$

In special geometry, the $\int_\Gamma \Omega_z$ is period. In the closed topological string, $\Gamma\in H_3(X,Z)$ is a 3-cycle. From the Stoke's theorem, $\mathrm{d}\beta_z$ term does not contribute. So the period $\int_\Gamma \Omega_z$ satisfies a homogenous Picard-Fuchs equation. While in open topological string, it is complicated for consideration of D-Branes and flux compactification. However, in the Ref. [12, 13], it was pointed out that the Γ must be a chain which satisfies $\partial\Gamma=C^+-C^-$ and its boundary $C^\pm$ is wrapped by D-branes. Thus, the open-string period $\int_\Gamma \Omega_z$ satisfies an inhomogeneous Picard-Fuchs equation. Following the work [12, 13], the domain wall tension of a D-brane wrapped on the chain Γ is proportional to the open-string period

$$\mathcal{T}_{\rm B}(z)=\int_\Gamma \hat{\Omega}(z), \tag{2}$$

here $\hat{\Omega}(z)$ is the normalization of the holomorphic 3-form $\Omega(z)$. So the inhomogeneous Picard-Fuchs equation is

$$L\mathcal{T}_{\rm B}(z)=f(z):=-\int_{\partial\Gamma}\beta, \tag{3}$$

which is essential for solving domain wall tension (superpotential) and various mathematical and physical data. S. T. Yau et al. [6] studied the superpotential in GKZ

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method, and they obtained a formula about the β. An explicit form of β for quintic in $P^4(1,1,1,1,1)$ was presented in Ref. [5] and that for hypersurfaces in weighted projective spaces were given in Ref. [10].

In this paper, we find that the form of β in the above formula is non-unique. Some attention should be paid to the different forms of β for two reasons. First, an appropriate form of the β can simplify the calculation of the D-brane domain wall tension. Second, if the cycle $C^{\pm}$ may have singularity, a good choice of the β form requires special attention. We also give another form of β for quintic in Section 2, and sextic in Section 3. Section 4 is for the other two models $Y^{(6)}$ and $Y^{(8)}$. The last section is a short summary.

2 The β of Picard-Fuchs equation for quintic

The one-parameter family of quintic 3-folds is given as

$$Y^{(5)}=\{[x_1,...,x_5]\in P^4|X=0\},$$

where

$$X:=\frac{1}{5}\left(x_1^5+x_2^5+x_3^5+x_4^5+x_5^5\right)-\psi x_1x_2x_3x_4x_5, \tag{4}$$

and $z=(5\psi)^{-5}$. The corresponding mirror manifold is resolutions of quotients of this one-parameter family of the quintic by Z_5^3. To derive the Picard-Fuchs equations, we use the 4-form on P^4,

$$\omega=\sum_i(-1)^{i-1}x_i\mathrm{d}x_1\wedge\ldots\wedge\widehat{\mathrm{d}x_i}\wedge\ldots\wedge\mathrm{d}x_5 \tag{5}$$

as well as the contraction of ω with the tangent vectors ∂_i $(i=1,\ldots,5)$

$$\omega_i=\omega(\partial_i). \tag{6}$$

A convenient choice of gauge for the holomorphic 3-form is [1, 5]

$$\Omega(z)=\mathrm{Res}_{X=0}\widetilde{\Omega}(z),\quad \text{where } \widetilde{\Omega}(z):=\frac{\omega}{X(z)}, \tag{7}$$

here Res means residue. The normalization of the holomorphic 3-form $\hat{\Omega}(z)$ is given as Ref. [5]

$$\hat{\Omega}(z)=\left(\frac{5}{2\pi\mathrm{i}}\right)^3\psi\Omega=\left(\frac{5}{2\pi\mathrm{i}}\right)^3\psi\mathrm{Res}_{X=0}\frac{\omega}{X}. \tag{8}$$

So the domain wall tension is

$$\mathcal{T}_{\mathrm{B}}(z)=\int_\Gamma\hat{\Omega}(z)=\int_{T_\varepsilon}\left(\frac{5}{2\pi\mathrm{i}}\right)^3\psi\widetilde{\Omega}(z), \tag{9}$$

where T_ε is a small tube around Γ. Notice

$$\partial_\psi^l\omega=l!\left(\frac{5}{2\pi\mathrm{i}}\right)^3\psi\int_{T_\varepsilon}\frac{(x_1x_2x_3x_4x_5)^l}{X^{l+1}}\omega. \tag{10}$$

For $l=4$ we can express it in combination of lower order derivatives by using the Griffiths-Dwork reduction

$$\mathrm{d}\left(\frac{A^i\omega_i}{X}\right)=\frac{\partial_iA^i\omega}{X}-\frac{A^i\partial_i\omega}{X^2}, \tag{11}$$

and the relation

$$\begin{aligned}&(1-\psi^5)(x_1^4x_2^4x_3^4x_4^4x_5^4)\\&=\psi^3x_1^2x_2^2x_3^2x_4^7x_5^3\partial_5X+\psi^4x_1^3x_2^3x_3^3x_4^4x_5^3\partial_4X\\&+\psi^2x_1x_2x_3^2x_4^6x_5^6\partial_3X+\psi x_2x_3^5x_4^5x_5^5\partial_2X\\&+x_2^4x_3^4x_4^4x_5^4\partial_1X.\end{aligned} \tag{12}$$

After some calculations, we can obtain another form of the β, denoted as $\beta^{(5)}$, for the quintic, as follows,

$$\begin{aligned}\beta^{(5)}=&\frac{1}{X}\left(\psi x_4\omega_4\right)+\frac{1}{X^2}\left(7\psi^2x_1x_2x_3x_4^2x_5\omega_4+\psi x_4^5x_5\omega_5\right)\\&+\frac{2}{X^3}\left(\psi x_3x_4^5x_5^5\omega_3+6\psi^3x_1^2x_2^2x_3^2x_4^3x_5^2\omega_4\right.\\&\left.+3\psi^2x_1x_2x_3x_4^6x_5^2\omega_5\right)+\frac{3!}{X^4}\left(x_2^4x_3^4x_4^4x_5^4\omega_1\right.\\&+\psi x_2x_3^5x_4^5x_5^5\omega_2+\psi^2x_1x_2x_3^2x_4^6x_5^6\omega_3\\&\left.+\psi^4x_1^3x_2^3x_3^3x_4^4x_5^3\omega_4+\psi^3x_1^2x_2^2x_3^2x_4^7x_5^3\omega_5\right),\end{aligned} \tag{13}$$

which is different from the following β form given in Ref. [5]

$$\begin{aligned}\beta=&\frac{1}{X}\left(\psi x_5\omega_5\right)+\frac{1}{X^2}\left(\psi x_4x_5^5\omega_4+7\psi^2x_1x_2x_3x_4x_5^2\omega_5\right)\\&+\frac{2}{X^3}\left(\psi x_3x_4^5x_5^5\omega_3+3\psi^2x_1x_2x_3x_4^2x_5^6\omega_4\right.\\&\left.+6\psi^3x_1^2x_2^2x_3^2x_4^2x_5^3\omega_5\right)+\frac{3!}{X^4}\left(x_2^4x_3^4x_4^4x_5^4\omega_1\right.\\&+\psi x_2x_3^5x_4^5x_5^5\omega_2+\psi^2x_1x_2x_3^2x_4^6x_5^6\omega_3\\&\left.+\psi^3x_1^2x_2^2x_3^2x_4^3x_5^7\omega_4+\psi^4x_1^3x_2^3x_3^3x_4^3x_5^4\omega_5\right),\end{aligned} \tag{14}$$

with the same method as Ref. [5], the inhomogeneous term $f(z)$ of Picard-Fuchs equation and the domain wall tension can be obtained as follows,

$$f(z)=\frac{15}{16\pi^2}\sqrt{z}, \tag{15}$$

and

$$\mathcal{T}_{\mathrm{B}}(z)=\frac{-4}{3}\sum_{m=0}^{\infty}\frac{\Gamma\left(-\frac{3}{2}-5m\right)}{\Gamma\left(-\frac{3}{2}\right)}\frac{\Gamma\left(\frac{1}{2}\right)^5}{\Gamma\left(\frac{1}{2}-m\right)^5}z^{-(m+\frac{1}{2})}, \tag{16}$$

which are consistent with the results in Ref. [5].

3 The β of Picard-Fuchs equation for sextic

The one-parameter Calabi-Yau hypersurface of degree $k=6$ in weighted projected space $P^4(1,1,1,1,2)$, sextic, is defined as

$$Y^{(6)}=\{[x_1,...,x_5]\in P^4(1,1,1,1,2)|W=0\}$$

where

$$W=\frac{1}{6}(x_1^6+x_2^6+x_3^6+x_4^6+2x_5^3)-\psi x_1x_2x_3x_4x_5, \tag{17}$$

and $z=4\cdot 6^{-6}\psi^{-6}$. The mirror manifold is resolution of quotients of this sextic by the group $G=G'/Z_6$, $G'=\mathrm{Ker}(\prod_{i=1}^{5}Z_{6/\nu_i}\rightarrow Z_6)$, where ν_i are the weights of coordinates $[x_1,...,x_5]$. Similarly, the 4-form is taken as Ref. [10]

$$\omega=\sum_i(-1)^{i-1}\nu_ix_i\mathrm{d}x_1\wedge...\wedge\widehat{\mathrm{d}x_i}\wedge...\wedge\mathrm{d}x_5. \tag{18}$$

The normalization of the holomorphic 3-form is [10]

$$\hat{\Omega}(z)=\frac{3\cdot 6^2}{(2\pi\mathrm{i})^3}\psi\Omega=\frac{3\cdot 6^2}{(2\pi\mathrm{i})^3}\psi\mathrm{Res}_{W=0}\frac{\omega}{W}. \tag{19}$$

By using the Griffiths-Dwork reduction, we obtain another form of the β, denoted as $\beta^{(6)}$ here,

$$\begin{aligned}\beta^{(6)}\ =\ &\frac{6}{W^4}(\psi^7x_1^3x_2^3x_3^3x_4^3x_5^4\omega_5+\psi^2x_1^4x_2^4x_3^4x_4^4x_5^2\omega_5\\&+\psi^6x_1^2x_2^2x_3^2x_4^3x_5^5\omega_4+\psi^3x_1^5x_2^5x_4^5x_5^2\omega_3+\psi^4x_1x_2^6x_4^6x_5^3\omega_2\\&+\psi^5x_1^2x_2x_3x_4^7x_5^4\omega_1)+\frac{2}{W^3}(\psi^4x_1^6x_4^6x_5\omega_5\\&+3\psi^6x_1^2x_2^2x_3^2x_4^2x_5^3\omega_5-2\psi x_1^3x_2^3x_3^3x_4^3x_5\omega_5\\&+2\psi^5x_1x_2x_3x_4^7x_5^2\omega_5+3\psi^6x_1^2x_2^2x_3^2x_4^3x_5^2\omega_4\\&+\psi^5x_1^2x_2x_3x_4^7x_5\omega_1)+\frac{1}{W^2}(x_1^2x_2^2x_3^2x_4^2\omega_5\\&+6\psi^5x_1x_2x_3x_4^2x_5\omega_4+\psi^4x_1^6x_4\omega_4\\&+\psi^5x_1^2x_2x_3x_4x_5\omega_1)+\frac{1}{W}\psi^4x_1\omega_1,\end{aligned} \tag{20}$$

which is different from that in Ref. [10], the inhomogeneous terms of Picard-Fuchs equation can be obtained by integrating the β on the curves given in Ref. [10], the result is

$$f(z)=\frac{3}{2\pi^2}\sqrt{z}, \tag{21}$$

the domain wall tension is consistent with the results in Ref. [10].

4 The other two models

The other two Calabi-Yau hypersurfaces are $Y^{(8)}$ and $Y^{(10)}$. They are very similar to sextic, for more details see Ref. [10]. We just list the last results.

$Y^{(8)}$ is defined as

$$Y^{(10)}=\{[x_1,...,x_5]\in P^4(1,1,1,1,4)|Y=0\}$$

where

$$Y=\frac{1}{8}(x_1^8+x_2^8+x_3^8+x_4^8+4x_5^2)-\psi x_1x_2x_3x_4x_5. \tag{22}$$

We obtain the β form, denoted as $\beta^{(8)}$ here, which is different from that in Ref. [10]

$$\begin{aligned}\beta^{(8)}\ =\ &\frac{6}{Y^4}[\psi^{10}x_1^3x_2^3x_3^3x_4^3x_5^4\omega_5+\psi^9x_1^2x_2^2x_3^2x_4^3x_5^4\omega_4+\psi^6x_1^7x_2^7x_4^7x_5\omega_3+\psi^7x_1^8x_2x_4^8x_5^2\omega_2+\psi^8x_1^2x_2x_3x_4^9x_5^3\omega_1+\psi^5x_1^6x_2^6x_3^6x_4^6x_5\omega_5\\&+\psi^4x_1^5x_2^5x_3^5x_4^5x_5^2\omega_5+\psi^3x_1^4x_2^4x_3^4x_4^4x_5^3\omega_5]+\frac{2}{Y^3}[3\psi^9x_1^2x_2^2x_3^2x_4^2x_5^3\omega_5+2\psi^8x_1x_2x_3x_4^9x_5^2\omega_5+2\psi^9x_1^2x_2^2x_3^2x_4^{10}x_5\omega_5\\&+2\psi^{10}x_1^3x_2^3x_3^3x_4^4x_5\omega_4-2\psi^{10}x_1^3x_2^3x_3^3x_4^3x_5^2\omega_5+\psi^7x_1^8x_4^8x_5\omega_5+\psi^8x_1^9x_2x_3x_4^2x_5\omega_4\\&+\psi^9x_1^3x_2^2x_3^2x_4^2x_5^2\omega_1-6\psi^2x_1^3x_2^3x_3^3x_4^3x_5^2\omega_5-3\psi^3x_1^4x_2^4x_3^4x_4^4x_5\omega_5-\psi^4x_1^5x_2^5x_3^5x_4^5\omega_5]\\&+\frac{1}{Y^2}[4\psi^8x_1x_2x_3x_4^2x_5\omega_4+2\psi^9x_1^2x_2^2x_3^2x_4^3\omega_4-6\psi^9x_1^2x_2^2x_3^2x_4^2x_5\omega_5+\psi^7x_1^8x_4\omega_4\\&+3\psi^8x_1^2x_2x_3x_4x_5\omega_1+15\psi x_1^2x_2^2x_3^2x_4^2x_5\omega_5+3\psi^2x_1^3x_2^3x_3^3x_4^3\omega_5]+\frac{1}{Y}[\psi^7x_1\omega_1-15x_1x_2x_3x_4\omega_5].\end{aligned} \tag{23}$$

$Y^{(10)}$ is defined as

$$Y^{(10)}=\{[x_1,...,x_5]\in P^4(1,1,1,5,2)|M=0\},$$

where

$$M=\frac{1}{10}(x_1^{10}+x_2^{10}+x_3^{10}+2x_4^5+5x_5^2)-\psi x_1x_2x_3x_4x_5. \tag{24}$$

The β form, denoted as $\beta^{(10)}$ here, is different from that in Ref. [10]

$$\begin{aligned}\beta^{(10)} =& \frac{6}{M^4}[\psi^{12}x_1^3x_2^3x_3^3x_4^3x_5^4\omega_5+\psi^{11}x_1^2x_2^2x_3^2x_4^3x_5^4\omega_4+\psi^{10}x_1x_2x_3^2x_4^6x_5^3\omega_3+\psi^9x_1x_3^{10}x_4^5x_5^2\omega_1+\psi^8x_1^9x_3^9x_4^4x_5\omega_2\\
&+\psi^7x_1^8x_2^8x_3^8x_4^3x_5\omega_5+\psi^6x_1^7x_2^7x_3^7x_4^2x_5^2\omega_5+\psi^5x_1^6x_2^6x_3^6x_4x_5^3\omega_5+\psi^4x_1^5x_2^5x_3^5x_5^4\omega_5+\psi^3x_1^4x_2^4x_3^4x_5^4\omega_4]\\
&+\frac{2}{M^3}[3\psi^{11}x_1^2x_2^2x_3^2x_4^2x_5^3\omega_5+2\psi^{10}x_1x_2x_3x_4^6x_5^2\omega_5+2\psi^{11}x_1^2x_2^2x_3^2x_4^3x_5^2\omega_4\\
&+\psi^9x_3^{10}x_4^5x_5\omega_5+\psi^{10}x_1x_2x_3^{11}x_4^2x_5\omega_4+\psi^{11}x_1^2x_2^2x_3^3x_4^2x_5^2\omega_3+7\psi^5x_1^6x_2^6x_3^6x_4x_5\omega_5\\
&-10\psi^2x_1^3x_2^3x_3^3x_4^3x_5^2\omega_5-10\psi^3x_1^4x_2^4x_3^4x_4^4x_5\omega_5-10\psi^4x_1^5x_2^5x_3^5x_4^5\omega_5-10\psi^5x_1^6x_2^6x_3^6x_4^2\omega_4-\psi^6x_1^7x_2^7x_3^7x_4^2\omega_5]\\
&+\frac{1}{M^2}[5\psi^{10}x_1x_2x_3x_4^2x_5\omega_4+2\psi^{10}x_1x_2x_3^2x_4x_5\omega_3+35\psi x_1^2x_2^2x_3^2x_4^2x_5\omega_5+\psi^9x_3x_4^5\omega_3\\
&+5\psi^2x_1^3x_2^3x_3^3\omega_5+5\psi^3x_1^4x_2^4x_3^4\omega_4+13\psi^4x_1^5x_2^5x_3^5\omega_5+10\psi^2x_1^3x_2^3x_3^3x_4^3\omega_5]+\frac{1}{M}[\psi^9x_4\omega_4-35x_1x_2x_3x_4\omega_5]. \qquad (25)\end{aligned}$$

The corresponding inhomogeneous terms and the related D-brane domain wall tensions computed by using these β forms, respectively, are consistent with the results in Ref. [10]

5 Summary

In this letter, we discuss the non-uniqueness of the β form in the inhomogeneous Picard-Fuchs equation, and give other forms of β for quintic and some one-parameter Calabi-Yau hypersurfaces in weighted projective spaces, respectively. Furthermore, the corresponding domain wall tensions are calculated by using these β forms and are consistent with the previous work. The β form is important, since different choices of β form not only show a kind of symmetry, but also provide a possibility for simplifying the calculation of the D-brane domain wall tension, etc. In particular, for some singular chains, the choice of β form requires special attention.

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