Recursive method for opacity expansion at finite temperature*

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Abstract: Using a reaction operator approach, we derive the multiple-scattering induced gluon number distribution function to all orders in powers of opacity at finite temperature. The detailed balance effect is analyzed by taking into account both gluon emission and absorption in a thermal medium. We also calculate virtual corrections and show that the infrared divergence cancels out in the gluon distribution function at finite temperature.

Key words: jet-quenching, non-abelian energy loss, opacity expansion, detailed balance

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1 Introduction

One of the most exciting phenomena observed at the RHIC is jet quenching, i.e., gluon radiation induced by multiple scattering for an energetic parton propagating in a dense medium.

Recent theoretical studies of parton energy loss have concentrated on gluon radiation induced by multiple scattering in a hot dense medium. Several approaches have been proposed to compute the medium-induced energy loss of a jet when it propagates in a dense medium. The opacity expansion method, developed by Widemann [1, 2], as well as Gyulassy, Lévai and Vitev (GLV) [3–6], has been used to study the non-Abelian energy loss. The inclusive gluon distribution to the nth order in opacity was derived at zero temperature, by using the reaction operator approach. This approach is based on the construction of a suitable reaction operator, $\hat{R}_n$, from which recursion relations for the inclusive gluon distribution can be derived and solved analytically at an arbitrary order n.

At finite temperature, since gluons are bosons, there should be stimulated gluon emission and absorption by the propagating parton due to the presence of thermal gluons in a hot dense medium. Such detailed balance is important for calculating parton energy loss [7].

In this paper, we follow the framework of opacity expansion and generalize the reaction operator approach to finite temperature cases. We calculate both real and virtual corrections and obtain the infrared safe inclusive gluon number distribution function to all orders in the opacity expansion.

2 Model and graphical shorthand

2.1 Model

The opacity is defined by the mean number of collisions in the medium, $\bar{n} \equiv L/\lambda = N\sigma/A_\perp$. Here N, L, and $A_\perp$ are the number, the thickness and the transverse area of the targets, and λ is the average mean-free path for the jet.

We employ the Gyulassy-Wang static color-screened Yukawa potential [8, 9] to model the interaction between the jet and target partons in a decon-

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fined quark-gluon plasma:

$$V_n = 2\pi\delta(q^0)v(\boldsymbol{q}_n)\mathrm{e}^{-\mathrm{i}\boldsymbol{q}_n\cdot\boldsymbol{x}_n}T_{a_n}(R)\otimes T_{a_n}(n), \quad (1)$$

where $\boldsymbol{x}_n$ is the location of the nth target parton and $v(\boldsymbol{q}_n)$ is given by

$$v(\boldsymbol{q}_n) = \frac{4\pi\alpha_s}{\boldsymbol{q}_n + \mu^2}, \quad (2)$$

and $T_{a_n}(R)$ and $T_{a_n}(n)$ are the color matrices for the jet and target parton respectively. The small transverse momentum transfer elastic cross section between the jet and target partons in this model is

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}(R,T)}{\mathrm{d}^2\boldsymbol{q}_\perp} = \frac{C_{\mathrm{R}}C_2(T)}{d_{\mathrm{A}}}\frac{|v(\boldsymbol{q}_\perp)|^2}{(2\pi)^2}, \quad (3)$$

where C_{R} is the casimir of the jet parton in the d_{R} dimensional representation, $C_2(T)$ is the casimir of the target parton in the d_{T} representation. C_{A} is the casimir in the adjoint gluon representation. The normalized distribution of momentum transer from the scattering centers is defined as follows:

$$|\bar{v}(\boldsymbol{q}_\perp)|^2 \equiv \frac{1}{\sigma_{\mathrm{el}}}\frac{\mathrm{d}^2\sigma_{\mathrm{el}}}{\mathrm{d}^2\boldsymbol{q}_\perp} = \frac{1}{\pi}\frac{\mu_{\mathrm{eff}}^2}{(\boldsymbol{q}_\perp^2+\mu^2)^2}, \quad (4)$$

where, in the Yukawa example, the normalization depends on the kinematic bounds through

$$\frac{1}{\mu_{\mathrm{eff}}^2} = \frac{1}{\mu^2} - \frac{1}{\boldsymbol{q}_{\perp\max}^2+\mu^2}. \quad (5)$$

and insure that $\int^{\boldsymbol{q}_{\perp\max}}\mathrm{d}^2\boldsymbol{q}_\perp\,|\bar{v}(\boldsymbol{q}_\perp)|^2 = 1$.

2.2 Graphical shorthand

A general graph consists of a gluon emission vertex, G_m, for emitting a gluon between centers z_m and z_{m+1}, and a specific set of direct interactions, X_{i,σ_i} for center i, and double Born interactions denoted as O_{j,a_j} for center j. The index $\sigma_i = 0,1$ stands for a single direct interaction at center i with the jet and gluon separately, while the index $a_j = 0,1,2$ denotes a contact interaction at center j with the jet, gluon and both jet+gluon, respectively.

In this notation, any diagram can be written in the form:

$$\mathcal{M} = \left[\prod_{i=0}^{m}T_{i,\alpha_i}\right]G_m\left[\prod_{j=m+1}^{n}T_{j,\beta_j}\right],$$

$$T_{i,\alpha_i}, T_{i,\beta_i} \in (X_{i,\sigma_i}, O_{i,a_i}). \quad (6)$$

While Eq. (6) can be used to enumerate all “single gluon emission with rescatterings” diagrams arising from a target with n aligned centers, it will prove convenient to group diagrams into classes of graphs that can be iteratively built from a diagrammatic kernel.

In this paper we employ the GLV formalism, which assumes that the parton is produced inside the medium (e.g., through $\mathrm{A}+\mathrm{A}\to\mathrm{q}+\bar{\mathrm{q}}+\mathrm{X}$, with high $Q^2 \equiv E^{+2}$) at a finite point $(t_0,\ \boldsymbol{x}_0)$ and then study the interaction between this outgoing jet and a hot medium: at this step, the gluon radiation before and after the multiple scattering between the jet and the medium parton are taken into consideration. The kernel for the hard production vertex is

$$\mathrm{Ker}^{(\mathrm{H})} = G_0 = -2\left(\frac{E-\omega}{E}\right)\mathrm{i}g_{\mathrm{s}}\frac{\boldsymbol{\epsilon}_\perp\cdot\boldsymbol{k}_\perp}{\boldsymbol{k}_\perp^2}\mathrm{e}^{\mathrm{i}\omega_0 z_0}c. \quad (7)$$

In the light-cone components,

$$k = [k^+,\ k^-,\ \boldsymbol{k}_\perp] = \left[2\omega,\ \frac{\boldsymbol{k}_\perp^2}{2\omega},\ \boldsymbol{k}_\perp\right]$$

is the four-momentum of the radiated gluon with polarization $\epsilon = [0, (\boldsymbol{\epsilon}_\perp\cdot\boldsymbol{k}_\perp)/k^+, \boldsymbol{\epsilon}_\perp]$, E is the initial jet energy, $|\omega|$ is the radiated (absorbed) gluon energy and the jet emerges with a momentum of

$$p = \left[2(E-\omega),\ \frac{\boldsymbol{k}_\perp^2}{2(E-\omega)},\ -\boldsymbol{k}_\perp\right].$$

c is the color of the radiated gluon and $\alpha_{\mathrm{s}} = g^2/4\pi$ is the strong coupling constant. We will also use the energy fraction $x \equiv |\omega|/E$ in the rest of the paper.

2.3 Construct diagrams

We focus here on the hard jet case relevant to nuclear collisions with a hard production vertex localized at z_0, as in Eq. (7). We consider the effect of final state interaction at position $z_i > z_0$ along the direction of the jet.

Let $\mathcal{A}$ denote a class of graphs with $N_{\mathcal{A}}$ members in which the last interaction has occurred at position $z_j < z_i$. We then enlarge this class of graphs. We can add a direct interaction at z_i and label it by $\mathcal{A}D_i$, where D_i specifies the direct insertion iteration. The new class contains $2N_{\mathcal{A}}$ diagrams:

1) $\mathcal{A}X_{i,0}$: A direct interaction with the jet

2) $\mathcal{A}X_{i,1}$: A direct interaction with the previously emitted gluon

In addition, there is a new special diagram, $\mathcal{A}G^{-1}X_{i,0}G_i = (\mathcal{A}D_i)_0$, where all interactions (direct or virtual), including the one at z_i, are with the jet and the gluon is emitted after all interactions at $z_i < z < \infty$. (G^{-1} amputates the gluon emission vertex of $\mathcal{A}_0$.)

The new class with $2N_{\mathcal{A}}+1$ graphs is constructed as

$$\mathcal{A} \Longrightarrow \mathcal{A}D_i = \mathcal{A}X_{i,0} + \mathcal{A}X_{i,1} + \mathcal{A}G^{-1}X_{i,0}G_i. \quad (8)$$

Similarly, we can consider the possibilities that arise from inserting a double Born virtual interaction at location z_i. This case includes a new subclass of diagrams among which one of the legs is attached to the jet line and the other one to the gluon line, i.e. $\mathcal{A}O_{i,2}$. So the new class $\mathcal{A}V_i$ has $3N_{\mathcal{A}}+1$ diagrams

$$\mathcal{A} \Longrightarrow \mathcal{A}V_i = \mathcal{A}O_{i,0}+\mathcal{A}O_{i,1}+\mathcal{A}O_{i,2}+\mathcal{A}G^{-1}O_{i,0}G_i. \quad (9)$$

3 Recursive method for opacity expansion

The GLV developed the recursive method, or reaction operator approach [3, 4], to calculate the induced gluon number distribution in the nth order in opacity at zero temperature. In this section, we generalize this method to finite temperature cases, in which one needs to take into account both stimulated gluon emission and absorption in a thermal medium with finite temperature T. Because of gluon absorption, we do not assume $E \gg |\omega|$ as in the zero temperature case. The phase space integration at finite temperature is given by

$$\mathrm{d}\varPhi = \frac{d^4k}{(2\pi)^3}\frac{1}{2|\boldsymbol{k}|}[(1+N(|\boldsymbol{k}|))\delta(\omega-|\boldsymbol{k}|) +N(|\boldsymbol{k}|)\delta(\omega+|\boldsymbol{k}|)], \quad (10)$$

where $N(|\boldsymbol{k}|)=1/[\exp(|\boldsymbol{k}|/T)-1]$ is the thermal gluon distribution.

In Sec. 2.3, we have discussed how to use the operator $\hat{D}_i$, $\hat{V}_i$ to construct new diagrams. Here we give the explicit formula. First, we recall the definitions of the Hard, Gunion-Bertsch and Cascade terms [10]

$$\boldsymbol{H}=\frac{\boldsymbol{k}_\perp}{\boldsymbol{k}_\perp^2}, \quad \boldsymbol{B}_i=\boldsymbol{H}-\boldsymbol{C}_i,$$
$$\boldsymbol{C}_{(i_1,i_2,\cdots,i_m)}=\frac{(\boldsymbol{k}_\perp-\boldsymbol{q}_{i_1\perp}-\boldsymbol{q}_{i_2\perp}-\cdots\boldsymbol{q}_{i_m\perp})}{(\boldsymbol{k}_\perp-\boldsymbol{q}_{i_1\perp}-\boldsymbol{q}_{i_2\perp}-\cdots\boldsymbol{q}_{i_m\perp})^2},$$
$$\boldsymbol{B}_{(i_1,i_2,\cdots,i_m)(j_1,j_2,\cdots,j_m)}=\boldsymbol{C}_{(i_1,i_2,\cdots,i_m)}-\boldsymbol{C}_{(j_1,j_2,\cdots,j_m)}. \quad (11)$$

Here and in the rest of the paper, we have suppressed the common factor $(2\mathrm{i}g_{\mathrm{s}}\boldsymbol{\epsilon}_\perp)$ in all diagrams for simplicity.

3.1 Amplitude iteration

3.1.1 General idea

In class $\mathcal{A}$ the sum of amplitudes can be denoted by

$$\mathcal{A}(x,\boldsymbol{k}_\perp,c)\equiv\sum_{\alpha}\mathcal{A}_{\alpha}(x,\boldsymbol{k}_\perp)\mathrm{Col}(c)_{\alpha} \quad (12)$$

where $\mathcal{A}_{\alpha}(x,\boldsymbol{k}_\perp)$ represents the kinematical part, $\mathrm{Col}(c)_{\alpha}$ stands for the color matrix for graphs in this class enumerated by α. Since by definition classes are constructed by repeated operations of either one of three operations, $\hat{1},\hat{D}_i,\hat{V}_i$, we can enumerate the 3^n different classes of diagrams via a tensor notation, $\mathcal{A}_{i_1\cdots i_n}$, where the indices $i_j=0,1,2$ specify whether there is no, a direct or a virtual interaction with the target parton at z_j:

$$\mathcal{A}_{i_1\cdots i_n}(x,\boldsymbol{k}_\perp,c) = \prod_{m=1}^{n}\left[\delta_{0,i_m}+\delta_{1,i_m}\hat{D}_m+\delta_{2,i_m}\hat{V}_m\right] \times G_0(x,\boldsymbol{k}_\perp,c). \quad (13)$$

The inclusive induced "probability" distribution at order n in the opacity expansion can be computed from the following sum of products over the 3^n classes that contribute at that order:

$$P_n(x,\boldsymbol{k}_\perp) = \bar{\mathcal{A}}^{i_1\cdots i_n}(c)\mathcal{A}_{i_1\cdots i_n}(c) \equiv \mathrm{Tr}\sum_{i_1=0}^{2}\cdots\sum_{i_n=0}^{2}\bar{\mathcal{A}}^{\dagger}_{i_1\cdots i_n}(x,\boldsymbol{k}_\perp,c) \times\mathcal{A}_{i_1\cdots i_n}(x,\boldsymbol{k}_\perp,c), \quad (14)$$

where the unique complementary class that contracts with $\mathcal{A}_{i_1\cdots i_n}$ is defined by

$$\bar{\mathcal{A}}^{i_1\cdots i_n}(x,\boldsymbol{k}_\perp,c) \equiv G_0^{\dagger}(x,\boldsymbol{k}_\perp,c)\prod_{m=1}^{n}\left[\delta_{0,i_m}\hat{V}_m^{\dagger}+\delta_{1,i_m}\hat{D}_m^{\dagger}+\delta_{2,i_m}\right]. \quad (15)$$

By the definitions of $\mathcal{A}$ and $\bar{\mathcal{A}}$, we can ensure that every $\bar{\mathcal{A}}\mathcal{A}$ product is in the same order separately.

With this tensor classification and construction, it becomes possible to construct P_n recursively from the lower rank (opacity) classes through the insertion of a "reaction" operator as follows:

$$P_n=\bar{\mathcal{A}}^{i_1\cdots i_{n-1}}\hat{R}_n\mathcal{A}_{i_1\cdots i_{n-1}}, \quad (16)$$

where

$$\hat{R}_n=\hat{D}_n^{\dagger}\hat{D}_n+\hat{V}_n+\hat{V}_n^{\dagger}. \quad (17)$$

Once we get the general solution for the gluon probability at the nth order in opacity, we can take the ensemble average over momentum transfers to obtain the induced gluon number distribution.

3.1.2 Construct $\hat{D}_n$ and $\hat{V}_n$

In this section we derive an explicit formula for $\hat{D}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c)$ and $\hat{V}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c)$, from which we can obtain the explicit form of Eq. (16) and then compute P_n recursively.

Using the same method as the GLV, we derive $\hat{D}_n$:

$$\hat{D}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c) = (a_n+\hat{S}_n+\hat{B}_n) \times\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c). \quad (18)$$

Here, we define two new operators: $\hat{S}_n$ and $\hat{B}_n$. $\hat{S}_n$, the "shift" or gluon scattering operator, is defined as

$$\hat{S}_n=\mathrm{i}f^{ca_nd}\times\mathrm{e}^{\mathrm{i}(\omega_0-\omega_n)z_n}\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{n\perp}\cdot\hat{\boldsymbol{b}}}. \quad (19)$$

where $\hat{\boldsymbol{b}} \equiv \mathrm{i}\overrightarrow{\nabla}_{\boldsymbol{k}\perp}$ is the impact parameter operator such that

$$\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{n\perp}\cdot\hat{\boldsymbol{b}}} f(\boldsymbol{k}_\perp) = f(\boldsymbol{k}_\perp - \boldsymbol{q}_{n\perp}). \tag{20}$$

The other operator $\hat{B}_n$ is defined as

$$\begin{aligned}&\hat{B}_n \mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c)\\ =& -\left(-\frac{1}{2}\right)^{N_\mathrm{v}(\mathcal{A}_{i_1\cdots i_{n-1}})}\left(\frac{E-\omega}{E}\right)\boldsymbol{B}_n\\ &\times \mathrm{e}^{\mathrm{i}\omega_0 z_n}[c,a_n]T_\mathrm{el}(\mathcal{A}_{i_1\cdots i_{n-1}}),\end{aligned} \tag{21}$$

where

$$\begin{aligned}&T_\mathrm{el}(\mathcal{A}_{i_1\cdots i_{n-1}}) \equiv (a_{n-1})^{i_{n-1}}\cdots(a_1)^{i_1},\\ &T^\dagger_\mathrm{el}(\bar{\mathcal{A}}^{i_1\cdots i_{n-1}}) \equiv (a_1)^{2-i_1}\cdots(a_{n-1})^{2-i_{n-1}}.\end{aligned} \tag{22}$$

$\left(-\frac{1}{2}\right)^{N_\mathrm{v}}$ in Eq. (21) arises because every virtual contact interaction introduces a factor $\left(-\frac{1}{2}\right)$ from the contact limit of the contour integration over longitudinal momentum. The numbers, N_v, $\bar{N}_\mathrm{v}$ of such contact interaction in the class $\mathcal{A}_{i_1\cdots i_{n-1}}$ and its complementary class $\bar{\mathcal{A}}^{i_1\cdots i_{n-1}}$ are defined as

$$\begin{aligned}N_\mathrm{v} &= N_\mathrm{v}(\mathcal{A}_{i_1\cdots i_{n-1}}) = \sum_{m=1}^{n-1}\delta_{2,i_m},\\ \bar{N}_\mathrm{v} &= N_\mathrm{v}(\bar{\mathcal{A}}^{i_1\cdots i_{n-1}}) = \sum_{m=1}^{n-1}\delta_{0,i_m}.\end{aligned} \tag{23}$$

According to Eq. (9), we can also derive the explicit form of $\hat{V}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c)$ by using a similar process as above for $\hat{D}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c)$. The result is

$$\begin{aligned}\hat{V}_n &= -\frac{1}{2}(C_\mathrm{A}+C_\mathrm{R}) - a_n\hat{S}_n - a_n\hat{B}_n\\ &= -a_n\hat{D}_n - \frac{1}{2}(C_\mathrm{A}-C_\mathrm{R}).\end{aligned} \tag{24}$$

3.2 Reaction operator recursion to all orders of opacity

In this section, we use the expressions for $\hat{D}_n$ and $\hat{V}_n$ derived above to obtain a general formula of P_n, with the help of Eq. (16).

Using Eqs. (18) and (24), we can re-express $\hat{R}_n$ in Eq. (17) as

$$\begin{aligned}\hat{R}_n =& \ (\hat{D}_n - a_n)^\dagger(\hat{D}_n - a_n) - C_\mathrm{A}\\ =& \ (\hat{S}_n+\hat{B}_n)^\dagger(\hat{S}_n+\hat{B}_n) - C_\mathrm{A}.\end{aligned} \tag{25}$$

Next, we insert Eq. (25) to Eq. (16), using the expressions of $\hat{S}_n$ in Eq. (19) and $\hat{B}_n$ in Eq. (21), we then derive the recursion relation of P_n:

$$\begin{aligned}P_n(\boldsymbol{k}_\perp) =& \ C_\mathrm{A}\left[P_{n-1}(\boldsymbol{k}_\perp - \boldsymbol{q}_{n\perp}) - P_{n-1}(\boldsymbol{k}_\perp)\right]\\ &-2C_\mathrm{A}\left(\frac{E-\omega}{E}\right)\boldsymbol{B}_n\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\omega_n z_n}\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{n\perp}\cdot\hat{\boldsymbol{b}}}\boldsymbol{I}_{n-1}\right]\\ &+\delta_{n,1}\left(\frac{E-\omega}{E}\right)^2 C_\mathrm{A}C_\mathrm{R}\left|\boldsymbol{B}_1\right|^2.\end{aligned} \tag{26}$$

We can solve Eq. (26) with the initial condition:

$$P_0 = G_0^\dagger G_0 = \left(\frac{E-\omega}{E}\right)^2 C_\mathrm{R}\left|\boldsymbol{H}\right|^2. \tag{27}$$

We thus obtain the general solution for gluon distribution probability at the nth order in opacity:

$$\begin{aligned}P_n(\boldsymbol{k}_\perp) =& -2\left(\frac{E-\omega}{E}\right)^2 C_\mathrm{R}C_\mathrm{A}^n\\ &\times\mathrm{Re}\sum_{i=1}^{n}\left[\prod_{j=i+1}^{n}(\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{j\perp}\cdot\hat{\boldsymbol{b}}}-1)\right]\boldsymbol{B}_i\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{i\perp}\cdot\hat{\boldsymbol{b}}}\mathrm{e}^{-\mathrm{i}\omega_0 z_i}\\ &\times\left[\prod_{m=1}^{i-1}(\mathrm{e}^{\mathrm{i}(\omega_0-\omega_m)z_m}\mathrm{e}^{\mathrm{i}\boldsymbol{q}_{m\perp}\cdot\hat{\boldsymbol{b}}}-1)\right]\\ &\times\boldsymbol{H}(\mathrm{e}^{\mathrm{i}\omega_0 z_1}-\mathrm{e}^{\mathrm{i}\omega_0 z_0}).\end{aligned} \tag{28}$$

This is the complete solution to the problem.

3.3 Ensemble average over momentum transfers

We now calculate the ensemble average over the scattering center location. Following the GLV [4], this reduces to an impact parameter average as follows:

$$\langle\cdots\rangle = \int\frac{\mathrm{d}^2\boldsymbol{b}}{A_\perp}. \tag{29}$$

It has been further shown in Ref. [4] that this leads to calculating:

$$\begin{aligned}&\langle P_n(\boldsymbol{k}_\perp)\rangle_\mathrm{v}\\ =& \int\prod_{m=1}^{n}[\mathrm{d}^2\boldsymbol{q}_{m\perp}\bar{v}_m^2(\boldsymbol{q}_{m\perp})]P_n(\boldsymbol{k}_\perp;\boldsymbol{q}_{1\perp}\cdots\boldsymbol{q}_{n\perp}).\end{aligned} \tag{30}$$

Inserting Eq. (26) in Eq. (30), we can obtain the recursion relation for the momentum transfer averaged probability for $n>1$ in the form

$$\begin{aligned}\langle P_n(\boldsymbol{k}_\perp)\rangle_\mathrm{v} =& \ C_\mathrm{A}\int\mathrm{d}^2\boldsymbol{q}_{n\perp}[\bar{v}_n^2(\boldsymbol{q}_{n\perp})-\delta^2(\boldsymbol{q}_{n\perp})]\times\langle P_{n-1}(\boldsymbol{k}_\perp-\boldsymbol{q}_{n\perp})\rangle_\mathrm{v}\\ &-2\left(\frac{E-\omega}{E}\right)^2 C_\mathrm{R}C_\mathrm{A}^n\int\prod_{i=1}^{n}[\mathrm{d}^2\boldsymbol{q}_{i\perp}(\bar{v}_i^2(\boldsymbol{q}_{i\perp})-\delta^2(\boldsymbol{q}_{i\perp}))]\\ &\times\boldsymbol{B}_n\cdot\boldsymbol{C}_{(1,\cdots,n)}\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\Phi_{n,n}}(\mathrm{e}^{\mathrm{i}\omega_{(1\cdots n)}(z_1-z_0)}-1)\right].\end{aligned} \tag{31}$$

$\Phi_{n,n}$ is the gluon elastic scattering phase shift from z_0 to z_n. The partial phase shift due to gluon rescattering from z_{m-1} to z_n is given by

$$\Phi_{n,m} = -\sum_{k=1}^{m} \omega_{(k\cdots n)}(z_k - z_{k-1}) = -\sum_{k=1}^{m} \omega_{(k\cdots n)}\Delta z_k. \tag{32}$$

With the value of P_1, we can obtain the solution for $n>0$ cases:

$$\langle P_n(\boldsymbol{k}_\perp)\rangle_{\rm v} = -2\left(\frac{E-\omega}{E}\right)^2 C_{\rm R}C_{\rm A}^n \times \int \prod_{i=1}^{n} [{\rm d}^2\boldsymbol{q}_{i\perp}(\bar{v}_i^2(\boldsymbol{q}_{i\perp}) - \delta^2(\boldsymbol{q}_{i\perp}))] \times \sum_{m=1}^{n} \boldsymbol{B}_{(m+1,\cdots,n)(m,\cdots,n)} \cdot \boldsymbol{C}_{(1,\cdots,n)} \times {\rm Re}\left[{\rm e}^{{\rm i}\Phi_{n,m}}({\rm e}^{{\rm i}\omega_{(1\cdots n)}(z_1-z_0)} - 1)\right]. \tag{33}$$

To obtain the final gluon number distribution, we need to multiply $\prod_j(\sigma_g(j)/A_\perp)$ along the path to convert $\bar{v}_j$ back into v_j and a combinational factor

$$\frac{N!}{n!(N-n)!} \approx \frac{N^n}{n!}. \tag{34}$$

that counts the number of ways n target partons out of N can be within the interaction range of the jet+gluon systems. Then after multiplying the phase space integral in Eq. (10), we finally obtain the induced gluon distribution:

$$x\frac{{\rm d}N^{(n)}}{{\rm d}x{\rm d}^2\boldsymbol{k}_\perp{\rm d}\omega} = \frac{C_{\rm R}\alpha_{\rm s}}{\pi^2}\left(\frac{E-\omega}{E}\right)^2\left(\frac{L}{\lambda_{\rm g}(1)}\right)^n\frac{1}{n!} \times \int\prod_{i=1}^{n}[{\rm d}^2\boldsymbol{q}_{i\perp}\left(\frac{\lambda_{\rm g}(1)}{\lambda_{\rm g}(i)}\right)(\bar{v}_i^2(\boldsymbol{q}_{i\perp}) - \delta^2(\boldsymbol{q}_{i\perp}))]$$
$$\times\left[(1+N(xE))\delta(\omega - xE) + N(xE)\delta(\omega + xE)\right] \times \left(-2\boldsymbol{C}_{(1,\cdots,n)}\sum_{m=1}^{n}\boldsymbol{B}_{(m+1,\cdots,n)(m,\cdots,n)}\right.$$
$$\left.\times\left[\cos\left(\sum_{k=2}^{m}\omega_{(k,\cdots,n)}\Delta z_k\right) - \cos\left(\sum_{k=1}^{m}\omega_{(k,\cdots,n)}\Delta z_k\right)\right]\right), \tag{35}$$

where

$$\omega_{(m\cdots n)} = \frac{E}{E-\omega}\frac{(\boldsymbol{k}_\perp - \boldsymbol{q}_{m\perp} - \cdots - \boldsymbol{q}_{n\perp})^2}{2\omega}. \tag{36}$$

Our $\omega_{(m\cdots n)}$ is different from the definition $\omega_{(m\cdots n)}^{\rm GLV}$ in GLV [4] by the extra factor $E/(E-\omega)$. Besides this difference in $\omega_{(m\cdots n)}$, our result for gluon number distribution is different from that of the GLV at zero temperature by an overall factor $(E-\omega)^2/E^2$ and the phase space integral. This simplicity in the transformation from zero-temperature to finite-temperature cases is likely due to the static form of the Yukawa potential.

In principle, one should also include virtual corrections to ensure unitarity and to obtain an infrared safe gluon number distribution. One can immediately see this is necessary since $\frac{{\rm d}N^{(n)}}{{\rm d}^2\boldsymbol{k}_\perp}$ is divergent when $|\boldsymbol{k}| \to 0$.

4 Virtual correction

As we have seen in the last section, infrared divergence appears in the inclusive gluon number distribution when we consider only real corrections. However, when one includes both real and virtual corrections, the infrared divergence will cancel out and we will obtain an infrared safe gluon number distribution. We will study the virtual corrections in this section.

We find that one could derive the virtual corrections by taking advantage of the result for real corrections. This is because one could generate virtual diagrams from the real processes in the following way: taking every real process diagram, we can make a virtual gluon by picking two gluon momenta k_1 and k_2, setting $k_1 = -k_2 = k$, multiplying by the gluon propagator and integrating over k, i.e.

$$\begin{aligned}{\rm d}N_{\rm vir.}^{(n)} =& {\rm i}T\sum_{n=-\infty}^{\infty}\int\frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\boldsymbol{k}^2} \\ &\times\left(\bar{\mathcal{A}}^{i_1\cdots i_n}(-k,c)\right)^\mu {\rm i}g_{\mu\nu}\left(\mathcal{A}_{i_1\cdots i_n}(k,c)\right)^\nu \\ =& \int\frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2|\boldsymbol{k}|}[1+2N(|\boldsymbol{k}|)]\bar{\mathcal{A}}^{i_1\cdots i_n}(-k,c) \\ &\times\mathcal{A}_{i_1\cdots i_n}(k,c).\end{aligned} \tag{37}$$

where ${\rm d}N_{\rm vir.}^{(n)}$ is the contribution of virtual gluon process to gluon distribution and

$$\begin{aligned}&\bar{\mathcal{A}}^{i_1\cdots i_n}(-x,-\boldsymbol{k}_\perp,c) \\ &\equiv G_0^\dagger(-x,-\boldsymbol{k}_\perp,c)\prod_{m=1}^{n}\left[\delta_{0,i_m}\hat{V}_m^\dagger + \delta_{1,i_m}\hat{D}_m^\dagger + \delta_{2,i_m}\right].\end{aligned} \tag{38}$$

Like before, we define the probability for virtual process

$$P_n^{\rm vir} = \bar{\mathcal{A}}^{i_1\cdots i_n}(-x,-\boldsymbol{k}_\perp,c)\mathcal{A}_{i_1\cdots i_n}(x,\boldsymbol{k}_\perp,c). \tag{39}$$

and we will also have the same recursion relation as

$$P_n^{\rm vir} = \bar{\mathcal{A}}^{i_1\cdots i_{n-1}}(-x,-\boldsymbol{k}_\perp,c)\hat{R}_n\mathcal{A}_{i_1\cdots i_{n-1}}(x,\boldsymbol{k}_\perp,c). \tag{40}$$

In the small x limit, we can construct the recursion relation between $P_n^{\rm vir}$ and $P_{n-1}^{\rm vir}$ exactly as before and derive the general formula for $P_n^{\rm vir}$ $(n>1)$ as follows:

$$\begin{aligned}P_n^{\rm vir} =& C_{\rm A}({\rm e}^{{\rm i}\boldsymbol{q}_{n\perp}\cdot\hat{\boldsymbol{b}}}-1)P_{n-1}^{\rm vir}+2C_{\rm R}C_{\rm A}^n\left(\frac{E^2-\omega^2}{E^2}\right)\boldsymbol{B}_n{\rm Re}[{\rm e}^{-{\rm i}\omega_n z_n}{\rm e}^{{\rm i}\boldsymbol{q}_{n\perp}\cdot\hat{\boldsymbol{b}}}]\\ &\times\left[\prod_{m=1}^{n-1}({\rm e}^{{\rm i}(\omega_0-\omega_m)z_m}{\rm e}^{{\rm i}\boldsymbol{q}_{m\perp}\cdot\hat{\boldsymbol{b}}}-1)\right]\times\boldsymbol{H}({\rm e}^{{\rm i}\omega_0 z_1}-{\rm e}^{{\rm i}\omega_0 z_0})\\ =& 2(1-x^2)C_{\rm R}C_{\rm A}^n{\rm Re}\sum_{i=1}^n\left[\prod_{j=i+1}^n({\rm e}^{{\rm i}\boldsymbol{q}_{j\perp}\cdot\hat{\boldsymbol{b}}}-1)\right]\times\boldsymbol{B}_i{\rm e}^{{\rm i}\boldsymbol{q}_{i\perp}\cdot\hat{\boldsymbol{b}}}{\rm e}^{-{\rm i}\omega_0 z_i}\\ &\times\left[\prod_{m=1}^{i-1}({\rm e}^{{\rm i}(\omega_0-\omega_m)z_m}{\rm e}^{{\rm i}\boldsymbol{q}_{m\perp}\cdot\hat{\boldsymbol{b}}}-1)\right]\boldsymbol{H}({\rm e}^{{\rm i}\omega_0 z_1}-{\rm e}^{{\rm i}\omega_0 z_0}).\end{aligned} \tag{41}$$

Then, following the same procedure for ensemble average:

$$\begin{aligned}\langle P_n^{\rm vir}\rangle_{\rm v} =& (1-x^2)C_{\rm R}C_{\rm A}^n\int\prod_{i=1}^n[{\rm d}^2\boldsymbol{q}_{i\perp}(\bar{v}_i^2(\boldsymbol{q}_{i\perp})-\delta^2(\boldsymbol{q}_{i\perp}))]\times\sum_{m=1}^n 2\boldsymbol{B}_{(m+1,\cdots,n)(m,\cdots,n)}\cdot\boldsymbol{C}_{(1,\cdots,n)}\\ &\times{\rm Re}\left[{\rm e}^{{\rm i}\Phi_{n,m}}({\rm e}^{{\rm i}\omega_{(1\cdots n)}(z_1-z_0)}-1)\right].\end{aligned} \tag{42}$$

Finally, we obtain the "induced gluon distribution" for virtual gluon process:

$$\begin{aligned}&x\frac{{\rm d}N_{\rm vir.}^{(n)}}{{\rm d}x{\rm d}^2\boldsymbol{k}_\perp{\rm d}\omega}\\ =&\frac{C_{\rm R}\alpha_{\rm s}}{\pi^2}(1-x^2)\delta(\omega)[1+N(xE)]\left(\frac{L}{\lambda_{\rm g}(1)}\right)^n\\ &\times\frac{1}{n!}\int\prod_{i=1}^n[{\rm d}^2\boldsymbol{q}_{i\perp}\left(\frac{\lambda_{\rm g}(1)}{\lambda_{\rm g}(i)}\right)(\bar{v}_i^2(\boldsymbol{q}_{i\perp})-\delta^2(\boldsymbol{q}_{i\perp}))]\\ &\times\left(2\boldsymbol{C}_{(1,\cdots,n)}\sum_{m=1}^n\boldsymbol{B}_{(m+1,\cdots,n)(m,\cdots,n)}\right.\\ &\times\left[\cos\left(\sum_{k=2}^m\omega_{(k,\cdots,n)}\Delta z_k\right)\right.\\ &\left.\left.-\cos\left(\sum_{k=1}^m\omega_{(k,\cdots,n)}\Delta z_k\right)\right]\right).\end{aligned} \tag{43}$$

One could easily verify that the infrared divergence cancels out between Eqs. (35) and (43). Thus, putting them together, we obtain an infrared safe gluon number distribution. Because of the existence of $\delta(\omega)$ in Eq. (43), the virtual process does not contribute to the effective energy loss. However, they are important to ensure unitarity and to obtain the infrared safe gluon distribution.

5 Conclusion

In summary, taking into account both gluon emission and absorption in a hot dense medium, we have derived the inclusive gluon number distribution to the nth order in opacity at finite temperature. We calculate both the real and virtual corrections, and show that the infrared divergence cancels out between them, thus producing an infrared safe gluon number distribution. In the future, we will use the results derived in this paper to calculate jet energy loss in a thermal medium and to find out how gluon absorption could change the behavior of the energy loss.

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