CPC(HEP & NP), 2010, **34**(3): 330–333 Chinese Physics C Vol. 34, No. 3, Mar., 2010

Topological structure of the solitons solution in $SU(3)$ Dunne-Jackiw-Pi-Trugenberger model*

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Abstract By using ϕ-mapping topological current theory and gauge potential decomposition, we discuss the self-dual equation and its solution in the $SU(N)$ Dunne-Jackiw-Pi-Trugenberger model and obtain a new concrete self-dual equation with a δ function. For the $SU(3)$ case, we obtain a new self-duality solution and find the relationship between the soliton solution and topological number which is determined by the Hopf index and Brouwer degree of ϕ-mapping. In our solution, the flux of this soliton is naturally quantized.

Key words Chern-Simons theory, Duune-Jackiw-Pi-Trugenberger model, topological number, soliton

PACS 11.15.-q, 02.40.-k

1 Introduction

Chern-Simons gauge fields play an important role in planar physics, it is a new type of gauge theory in two dimensions [1]. Those Chern-Simons theories are interesting both for their theoretical novelty and practical applications such as the quantum Hall effect in condensed matter physics [2] and the fractional spin in quantum field theory [3]. Chern-Simons term acquires dynamics via coupling to other fields [4], and gets multifarious gauge theory. Nonrelativistic Chern-Simons theory supports solitons solution. These static solutions can be obtained when their Hamiltonian is minimal. R. Jackiw and S. Y. Pi considered a gauged, nonliner Schödinger equation in two spatial dimensions, which describes nonrelativistic matter interacting with Chern-Simons gauge fields [4, 5]. Then Dunne et al found that the nonliner Schödinger equation with additional coupling to non-Abelian Chern-Simons gauge fields also possesses static, zero-energy solutions which satisfy self-dual equations [6], they encountered various well-known nonlinear equations of two dimentional physics with spatial Ansätze for the Lie algebraic structures of $SU(N)$. This model is called the Dunne-Jackiw-Pi-Trugenberger model (DJPT model). In our former work, we have studied the topological structure of solitons solutions in the Jakiw-Pi model and $SU(2)$ DJPT model [7–9].

In this paper, by using gauge potential decomposition and ϕ-mapping theory, we will discuss the topological structure of the self-dual solution in the DJPT model. We will look for a new self-dual equation and complete soliton solution of the $SU(3)$ DJPT model, and set up the relationship between the soliton solution and topological number which is determined by the Hopf index and Brouwer degree. We also study the quantization of the flux of the soliton.

2 The Toda equation with a δ function

The nonrelativistic self-dual Chern-Simons system describes charged scalar fields Ψ with nonrelativistic dynamics, and this system is a 2+1 dimensional space-time model, which minimally coupled to gauge fields A_μ with Chern-Simons dynamics [6]. Its Lagrange density is

$$\mathcal{L} = -\kappa\mathcal{L}_{cs} + \mathrm{itr}(\Psi^\dagger D_0\Psi) - \frac{1}{2m}\mathrm{tr}((D_i\Psi)^\dagger D_i\Psi) + \frac{1}{4m\kappa}\mathrm{tr}\big([\Psi,\Psi^\dagger]^2\big), \quad (1)$$

Received 22 April 2009, Revised 28 August 2009
* Supported by Talent Introduction Project of Xianyang Normal University (07XSYK217)
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where Ψ is the matter field matrix and the matter density reads

$$\rho=\mathrm{i}[\Psi,\Psi^\dagger], \tag{2}$$

so the magnetic field is

$$B=-\frac{1}{\kappa}\rho, \tag{3}$$

the $\mathcal{L}_{\rm cs}$ in Eq. (1) is Chern-Simons Lagrange density

$$\mathcal{L}_{\rm cs}=\epsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho+\frac{2}{3}A_\mu A_\nu A_\rho\right). \tag{4}$$

The energy density can be written as

$$\varepsilon=\frac{1}{2m}\mathrm{tr}\big((D_{\rm i}\Psi)^\dagger D_{\rm i}\Psi\big), \tag{5}$$

when the energy density is minimized, the non-relativistic self-dual Chern-Simons equations have static solutions, the self-dual equations are [6]

$$\partial_\pm\Psi+[A_\pm,\Psi]=0,\quad \partial_\mp\Psi^\dagger+[A_\mp,\Psi^\dagger]=0, \tag{6}$$

$$\partial_\pm A_\mp-\partial_\mp A_\pm+[A_\pm,A_\mp]=\pm\frac{2}{\kappa}[\Psi,\Psi^\dagger], \tag{7}$$

here using the notation $A_\pm$ for $A_x\pm\mathrm{i}A_y$ and $\partial_\pm$ for $\partial_x\pm\mathrm{i}\partial_y$, for definiteness, but without loss of generality, we shall take $\kappa>0$ and lower signs, so the self-dual equation becomes

$$\partial_-\Psi+[A_-,\Psi]=0,\partial_+\Psi^\dagger+[A_+,\Psi^\dagger]=0, \tag{8}$$

$$\partial_- A_+-\partial_+ A_- -[A_+,A_-]=\frac{2}{\kappa}[\Psi^\dagger,\Psi]. \tag{9}$$

Suppose the fields have the following Lie algebra decomposition

$$\Psi=\sum_{\alpha=1}^{r}u_\alpha\mathrm{e}^\alpha\ , \tag{10}$$

$$A_-=\sum_{\alpha=1}^{r}A_\alpha h^\alpha\ , \tag{11}$$

$$A_+=-\sum_{\alpha=1}^{r}A^*_\alpha h^\alpha\ , \tag{12}$$

in which h^α and e^α are the Chevalley basis of the $SU(N)$ Lie algebra, and $r=N-1$ for $SU(N)$ case. We consider only the simply-laced algebras for easy presentation:

$$[h^\alpha,\mathrm{e}^\beta]=K_{\beta\alpha}\mathrm{e}^\beta, \tag{13}$$

$$[\mathrm{e}^\alpha,\mathrm{e}^{-\beta}]=\delta_{\beta\alpha}h^\alpha, \tag{14}$$

where K is nonsingular matrix and called Cartan matrix. For $SU(N)$ the Cartan matrix is $(N-1)\times(N-1)$, with all diagonal equal to 2 and -1 entered above, the matrix will be

$$K=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2 & -1 & 0\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \tag{15}$$

substituting Eqs. (10)–(12) into Eq. (8) and Eq. (9), we can obtain

$$\partial_-u_\alpha+u_\alpha\sum_{\beta=1}^{r}K_{\alpha\beta}A_\beta=0,\quad \partial_+u^\dagger_\alpha+u^\dagger_\alpha\sum_{\beta=1}^{r}K_{\alpha\beta}A^*_\beta=0, \tag{16}$$

those equations requires that

$$\partial_-A^*_\alpha+\partial_+A_\alpha=\frac{2}{\kappa}\,|\,u_\alpha\,|^2\,. \tag{17}$$

We note that when u_α is decomposed into two scalar fields

$$u_\alpha=u^1_\alpha+\mathrm{i}u^2_\alpha, \tag{18}$$

a unit vector field $\boldsymbol{n}_\alpha$ is defined as

$$n^a_\alpha=\frac{u^a_\alpha}{\sqrt{\rho_\alpha}},a=1,2, \tag{19}$$

where ρ_α is the matter density component which is defined by $\rho_\alpha\equiv u^*_\alpha u_\alpha$. It is easy to prove that $\boldsymbol{n}$ satisfies the constraint conditions

$$n^1_\alpha n^1_\alpha+n^2_\alpha n^2_\alpha=1;\quad n^1_\alpha\mathrm{d}n^1_\alpha+n^2_\alpha\mathrm{d}n^2_\alpha=0. \tag{20}$$

The A_α in Eq. (11) and Eq. (12) can be wriiten as follows

$$A_\alpha=A^1_\alpha+\mathrm{i}A^2_\alpha;\quad A^\star_\alpha=A^1_\alpha-\mathrm{i}A^2_\alpha, \tag{21}$$

substituting Eq. (18) into Eq. (16), and using Eq. (19) and Eq. (21), we get

$$-\sum_{\beta=1}^{r}K_{\alpha\beta}A^i_\beta=\sum_{a,b=1}^{2}\epsilon^{ab}n^a_\alpha\partial_i n^b_\alpha-\frac{1}{2}\epsilon^{ij}\partial_j\ln\rho_\alpha, \tag{22}$$

where the decomposition of $U(1)$ gauge potential has been used [10]. In Eq. (22) $i,j=1,2$ and using the symbols $\partial_1=\partial_x$ and $\partial_2=\partial_y$. Substituting Eq. (21) and Eq. (22) into Eq. (17), and making use of the ϕ-mapping topological current theory [11], we can get

$$\nabla^2\ln\rho_\alpha=-\frac{2}{\kappa}\sum_{\beta=1}^{r}\rho_\beta-4\pi\delta^2(\boldsymbol{u}_\alpha)J\left(\frac{\boldsymbol{u}_\alpha}{\boldsymbol{r}}\right), \tag{23}$$

here $\boldsymbol{u}_\alpha=(u^1_\alpha,u^2_\alpha)$, $\boldsymbol{r}=(x,y)$, and the $J\left(\frac{\boldsymbol{u}_\alpha}{\boldsymbol{r}}\right)$ is Jacobian

$$J\left(\frac{\boldsymbol{u}_\alpha}{\boldsymbol{r}}\right)=\frac{1}{2}\sum_{a,b,i,j=1}^{2}\epsilon^{ab}\epsilon^{ij}\partial_i u^a_\alpha\partial_j u^b_\alpha, \tag{24}$$

it is obvious that when $\rho_\alpha \neq 0$, this Eq. (23) will be the Toda equation [12], and the δ function describes the singular point of $\rho_\alpha = 0$.

3 Topological structure and flux of the soliton solution in the $SU(3)$ DJPT model

In this section we will study the Toda equation with a δ function, for $SU(2)$, the toda equation will be the Liouville equation, the solution of this equation is like $U(1)$ case [9].

For $SU(3)$, Eq. (23) gives two equations

$$\nabla^2 \ln\rho_1 = -\frac{2}{\kappa}(2\rho_1-\rho_2) - 4\pi\delta^2(\boldsymbol{u}_1)J\left(\frac{\boldsymbol{u}_1}{\boldsymbol{r}}\right), \quad (25)$$

$$\nabla^2 \ln\rho_2 = -\frac{2}{\kappa}(2\rho_2+\rho_1) - 4\pi\delta^2(\boldsymbol{u}_2)J\left(\frac{\boldsymbol{u}_2}{\boldsymbol{r}}\right), \quad (26)$$

when $\rho \neq 0$, Eq. (25) and Eq. (26) will be

$$\nabla^2 \ln\rho_1 = -\frac{2}{\kappa}(2\rho_1-\rho_2), \quad (27)$$

$$\nabla^2 \ln\rho_2 = -\frac{2}{\kappa}(2\rho_2+\rho_1), \quad (28)$$

that are solved by [6]

$$\rho_1 = \frac{\kappa}{2}\nabla^2 \ln(1+|\varphi_1|^2 + \frac{1}{4}|\varphi_1\varphi_2+\Phi|^2), \quad (29)$$

$$\rho_2 = \frac{\kappa}{2}\nabla^2 \ln(1+|\varphi_2|^2 + \frac{1}{4}|\varphi_1\varphi_2-\Phi|^2), \quad (30)$$

where φ_1, φ_1 and Φ depend only on z with

$$\Phi' = \varphi_1'\varphi_2 - \varphi_1\varphi_2', \quad (31)$$

a convenient choice is

$$\varphi_1 = \left(\frac{z}{z_0}\right)^{N_1}, \quad \varphi_2 = \left(\frac{z}{z_0}\right)^{N_2}, \quad (32)$$

then

$$\Phi = \frac{N_1-N_2}{N_1+N_2}\left(\frac{z}{z_0}\right)^{N_1+N_2}, \quad (33)$$

so the density is

$$\rho_1 = \frac{\kappa}{2}\nabla^2 \ln\left(1+\left(\frac{r}{r_0}\right)^{2N_1} + \left|\frac{N_1}{2(N_1+N_2)}\right|\left(\frac{r}{r_0}\right)^{2(N_1+N_2)}\right), \quad (34)$$

$$\rho_2 = \frac{\kappa}{2}\nabla^2 \ln\left(1+\left(\frac{r}{r_0}\right)^{2N_2} + \left|\frac{N_2}{2(N_1+N_2)}\right|\left(\frac{r}{r_0}\right)^{2(N_1+N_2)}\right). \quad (35)$$

Under this radially symmetric situation, $\nabla^2 \ln\rho$ can be expressed as

$$\nabla^2 \ln\rho = \frac{\partial^2}{\partial r^2}\ln\rho + \frac{1}{r}\ln\rho, \quad (36)$$

integrating Eq. (23)

$$\int \nabla^2 \ln\rho_1 \mathrm{d}\boldsymbol{r} = -\int \frac{\kappa}{2}(2\rho_1-\rho_2) + 4\pi\delta^2(\boldsymbol{u}_1)J\left(\frac{\boldsymbol{u}_1}{\boldsymbol{r}}\right)\mathrm{d}\boldsymbol{r}, \quad (37)$$

in order to investigate the density on single point $r=0$, we only calculate

$$\lim_{r\to 0}\int \nabla^2 \ln\rho_1 \mathrm{d}\boldsymbol{r} = -\lim_{r\to 0}\int 4\pi\delta^2(\boldsymbol{u}_1)J\left(\frac{\boldsymbol{u}_1}{\boldsymbol{r}}\right)\mathrm{d}\boldsymbol{r}, \quad (38)$$

the left side of this equation is

$$\lim_{r\to 0}\int \nabla^2 \ln\rho_1 \mathrm{d}\boldsymbol{r} = 4\pi(N_1-1), \quad (39)$$

and the right side is

$$-\lim_{r\to 0}\int 4\pi\delta^2(\boldsymbol{u}_1)J\left(\frac{\boldsymbol{u}_1}{\boldsymbol{r}}\right)\mathrm{d}\boldsymbol{r} = -4\pi Q_1, \quad (40)$$

from Eq. (39) and Eq. (40) we can obtain

$$N_1 = -Q_1 + 1, \quad (41)$$

here Q_1 is the topological number of the soliton,

$$Q_1 = \beta_1\eta_1, \quad (42)$$

where β_1 is the positive integer (the Hopf index of the zero point) and $\eta_1 = \pm 1$, the Brouwer degree of the vector field $\boldsymbol{u}_1$ [13].

And in the same way

$$N_2 = -Q_2 + 1, \quad (43)$$

Eq. (41) and Eq. (43) can be written in one equation

$$N_\alpha = -Q_\alpha + 1, \alpha = 1, 2, \quad (44)$$

from here we can see that N_α must be an integer.

The matter density is

$$\rho = \mathrm{i}[\Psi, \Psi^\dagger] = \mathrm{i}\rho_1 h^1 + \mathrm{i}\rho_2 h^2, \quad (45)$$

the magnetic field B is

$$B = -\frac{1}{\kappa}\rho, \quad (46)$$

so the magnetic flux of this soliton is

$$\Phi = \int_0^\infty B\mathrm{d}\boldsymbol{r} = 2\pi\mathrm{i}(Q_1+Q_2-2)(h^1+h^2), \quad (47)$$

in which h^1 and h^2 are the Chevalley basis of the $SU(3)$ Lie algebra, and this equation indicates that the magnetic flux is quantized.

4 Summary and concluding remarks

When the nonliner Schödinger equation is coupled to Chern-Simons fields, it gives static, zero-energy solutions that satisfy self-dual equations, the solutions correspond to solitons and vortices. In this paper, we discuss the Toda equation of DJPT model with gauge potential decomposition and ϕ-mapping theory, and find a new self-dual equation (Eq. (23)) which is a Toda equation with a δ function, when $\rho_\alpha \neq 0$, Eq. (23) will be the Toda equation, and the δ function describes the singular point of $\rho_\alpha = 0$. In Sec. 3, we solve Eq. (23) when the gauge Lie algebraic is $SU(3)$ and find the parameters N_1 and N_2 of the solution are determined by the topological number of the soliton, so the flux of the soliton is quantized.

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