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Classical mechanics in non-commutative phase space*

WEI Gao-Feng(卫高峰)[1,2;1)] LONG Chao-Yun(龙超云)[1,2] LONG Zheng-Wen(隆正文)[1,2]
QIN Shui-Jie(秦水介)[1] FU Qiang(付强)[3]

1 (Laboratory for Photoelectric Technology and Application, Guizhou University, Guiyang 550025, China)
2 (Department of Physics, College of Science, Guizhou University, Guiyang 550025, China)
3 (Xi'an Technological University, Xi'an 710032, China)

Abstract In this paper the laws of motion of classical particles have been investigated in a non-commutative phase space. The corresponding non-commutative relations contain not only spatial non-commutativity but also momentum non-commutativity. First, new Poisson brackets have been defined in non-commutative phase space. They contain corrections due to the non-commutativity of coordinates and momenta. On the basis of this new Poisson brackets, a new modified second law of Newton has been obtained. For two cases, the free particle and the harmonic oscillator, the equations of motion are derived on basis of the modified second law of Newton and the linear transformation (Phys. Rev. D, 2005, **72**: 025010). The consistency between both methods is demonstrated. It is shown that a free particle in commutative space is not a free particle with zero-acceleration in the non-commutative phase space, but it remains a free particle with zero-acceleration in non-commutative space if only the coordinates are non-commutative.

Key words non-commutative geometry, classical mechanics, free particle, harmonic oscillator

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1 Introduction

In the past few years problems in non-commutative spaces have attracted much attention[1—4]. This interest arose while studying the problem of open strings attached to a D-brane in the presence of a non-vanishing background B-field, inducing non-commutativity in its end points[5—8], and studying the Hall effect[9], showing non-commutativity in the canonical coordinates and momenta. One way to deal with the non-commutative space is to construct a new kind of field theory by changing the standard product of the fields by the star product (Weyl-Moyal):

$$(f*g)(x)=\exp\left(\frac{\mathrm{i}}{2}\theta_{ij}\,\partial_i\,\partial_j\right)f(x)g(y)\big|_{x=y} \ . \qquad (1)$$

Here the constant parameters θ_{ij}, which are real and anti-symmetric matrix elements, represent the non-commutativity of the space; f and g are infinitely differentiable functions. In this theory some interesting results have been found[10, 11]. Another approach is, to postulate the commutation relations[14]:

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\hbar\theta_{ij},\quad [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij},\quad [\hat{p}_i,\hat{p}_j]=0. \qquad (2)$$

In this case a non-commutative quantum mechanics can be formulated from which some relevant results have already been obtained[12, 13]. Subsequently, in another interesting research by Juan M. Romero et al[14], it was assumed that the phase space has a symplectic structure consistent with the commutation rules of the non-commutative quantum mechanics. But Juan M. Romero et al. only derived the corrections due to the non-commutativity of the coordinates. As a result a modification of Newtons second law has been obtained. In fact, although in string theory only the coordinate space exhibits a non-commutative structure, considering the momentum being the partial derivatives of the action with respect to the non-commutative spatial coordinates, naturally, momentum space also exhibits a non-commutative structure[15]. Although the effects of non-commutativity should presumably be-

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1) E-mail: fgwei_2000@163.com

come significant at very high energy scales close, for instance, to the string scale, it is expected that there should be some relics of the effects of spatial non-commutativity because of the incomplete decoupling mechanism between the high and low energy sectors[15, 16]. The quantum mechanics on non-commutative spaces seems to be able to reveal such low energy relics[15, 16] and has thus been investigated intensively. Some significant results have been obtained, for example, quantum Hall effect[9], quantum gravity field[15] etc. It is well known that classical Poisson brackets should go over to the commutation relations of quantum mechanics via Eq. (4), namely, interpreting classical mechanics as the classical approximation to quantum mechanics. So we believe that the theoretical study of non-commutative classical algebraic relation is also significant, for instance, in the non-commutative second law of Newton[14] etc. In this paper we present new corrections to Newton's second law for the case when the coordinates and momenta are all non-commutative. First, we redefine Poisson brackets of Ref. [14] and give the corresponding new corrections to Newton's second law in section 2. Second, on the basis of new modified second law of Newton, we derive the equation of motion for the free particle in Section 3. Finally, taking the harmonic oscillator as an example, the equation of motion obtained, based on the new modified second law of Newton, is consistent with the one obtained by linear transformations[15] in Section 4. Interestingly, it is shown that a free particle in commutative space isn't a free particle with zero-acceleration in the non-commutative phase space. But it is also free particle with zero-acceleration in non-commutative space when only the coordinates are non-commutative. The conclusion is given in Section 5.

2 Non-commutative classical mechanics

In order to describe a non-commutative phase space, the commutation relations in Eq. (2) should be changed as follows[15]:

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\hbar\theta_{ij} \ ,$$

$$[\hat{p}_i,\hat{p}_j]=\mathrm{i}\hbar\eta_{ij} \ ,$$

$$[\hat{x}_i,\hat{p}_j]=\hbar_{\text{eff}}\delta_{ij} \quad i,j=1,2,$$

$$\hbar_{\text{eff}}=\hbar(1+\xi)\,, \quad \xi=\frac{\mathrm{Tr}[\theta\eta]}{4\hbar^2} \ , \tag{3}$$

where θ_{ij}, η_{ij} are real and anti-symmetric matrix elements, representing the non-commutativity of coordinates and momenta respectively.

In the classical limit the quantum mechanical commutator is replaced by the Poisson bracket via

$$\frac{1}{\mathrm{i}\hbar}[\hat{A},\hat{B}]\rightarrow\{\tilde{A},\tilde{B}\} \ . \tag{4}$$

In the following discussion $\tilde{F}$ denotes the variables in the non-commutative phase space in order to distinguish them from the variables F in the commutative space. Noting that ξ is of second order on the non-commutative parameters θ and η, (namely $\xi\ll 1$), the classical limit of Eq. (3) reads

$$\{\tilde{x}_i,\tilde{x}_j\}=\theta_{ij}, \quad \{\tilde{p}_i,\tilde{p}_j\}=\eta_{ij}, \quad \{\tilde{x}_i,\tilde{p}_j\}=\delta_{ij} \ . \tag{5}$$

As mentioned in Refs. [17, 18], the Poisson brackets must possess the same properties as the quantum mechanical commutators; namely, they must be bilinear, anti-symmetric, satisfy the Leibniz rules and the Jacobi identity. The general form of the Poisson brackets for this deformed version of classical mechanics can be redefined as (the correctness is proven in section 4):

$$\{A,B\} = \left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_j}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_j}\right)\{x_i,p_j\}+ \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_j}\{x_i,x_j\}+\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_j}\{p_i,p_j\} \ . \tag{6}$$

Substituting Eq. (5) into Eq. (6), the redefined Poisson brackets can be written as follows:

$$\{A,B\} = \left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i}\right)+ \theta_{ij}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_j}+\eta_{ij}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_j} \ . \tag{7}$$

Now let us consider a two-dimensional Hamiltonian of the form

$$H=\frac{1}{2m}(p_1^2+p_2^2)+V(x_1,x_2) \ . \tag{8}$$

The equations of motion corresponding to this symplectic structure are given by

$$\dot{x}_i=\{x_i,H\}=\frac{p_i}{m}+\theta_{ij}\frac{\partial V}{\partial x_j} \ , \tag{9}$$

$$\dot{p}_i=\{p_i,H\}=-\frac{\partial V}{\partial x_i}+\eta_{ij}\dot{x}_j \ , \tag{10}$$

which can be written as

$$m\ddot{x}_i=-\frac{\partial V}{\partial x_i}+\eta_{ij}\dot{x}_j+\theta_{ij}m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial x_j}\right), \quad i=1,2. \tag{11}$$

We interpret these equations as the new second law of Newton. The second term of Eq. (11) is a correction due to the non-commutativity of momentum. The third term is the correction due to the non-commutativity of coordinates. These new terms depend on the background, space through the factor of "non- commutativity" θ_{ij} and η_{ij}, but also through the variations in the potential. The external fields produce a perturbation in the space, inducing this way these new forces. For $\eta_{ij}=0$, Eq. (11) goes over into the results of Ref. [14], which therefore only discusses a special case of Eq. (11).

3 Equation of motion of a free particle in non-commutative phase space

Taking a two-dimensional free particle as an example, we have

$$V(x_1,x_2)=0 \ . \tag{12}$$

The substitution of Eq. (12) into Eq. (11) gives

$$m\ddot{x}_i=\eta_{ij}\dot{x}_j \ . \tag{13}$$

This is the new equation of motion for the free particle in non-commutative phase space. Surprisingly, Eq. (13) shows that the acceleration of a free particle in non-commutative phase space is non-zero. The term on the right hand of Eq. (13) is an analogue to a damping force caused by non-commutativity of the momentum components. We conclude that a free particle in commutative space isn't a free particle with zero-acceleration in the non-commutative phase space. The reason for this non-vanishing acceleration is solely the non-commutativity of momentum components. It remains, however, a free particle in a non-commutative space where only the coordinates are non-commutative.

4 Equation of motion of a harmonic oscillator in non-commutative phase space

The harmonic oscillator model plays an important role not only in classical quantum mechanics, but also in non-commutative quantum mechanics. It has been subject to many investigations[19—23]. In order to prove the correctness of the redefined Poisson brackets and new corrections to Newton's second law, let us take the two-dimensional harmonic oscillator as an example.

$$V(x_1,x_2)=\frac{1}{2}k\sum_{i=1}^{2}x_i^2 \ . \tag{14}$$

The substitution of Eq. (14) into Eq. (11) gives

$$m\ddot{x}_i=-kx_i+\eta_{ij}\dot{x}_j+\theta_{ij}mk\dot{x}_j \ . \tag{15}$$

This is the new second law of Newton for the harmonic oscillator. The fist term on the right-hand side of Eq. (15) is the elasticity, the second and third terms are the analogues of the damping force due to the non-commutativity of momenta and coordinates respectively.

It is worth noting that the Eqs. (13) and (15) both show correlations between different degrees of freedom, i.e. the acceleration degree of freedom i is correlated with the velocity degree of freedom j in non-commutative phase space, and this correlation is dependent on non-commutative parameters.

Finally, in order to prove the correctness of the redefined Poisson brackets and Newton's modified second law, we will solve the equation of motion of the harmonic oscillator by linear transformations[15] in the non-commutative phase space.

The Hamiltonian of a harmonic oscillator in non-commutative phase space has the following form:

$$H=\frac{1}{2m}(\tilde{p}_1^2+\tilde{p}_2^2)+V(\tilde{x}_1,\tilde{x}_2) \ . \tag{16}$$

where the non-commutative phase space variables $\{\tilde{x}_1,\tilde{p}_1,\tilde{x}_2,\tilde{p}_2\}$ satisfy the commutation relations of Eq. (5).

According to Ref. [15], one possible way of implementing the algebra of Eq. (5) is to construct the non-commutative variables $\{\tilde{x}_1,\tilde{p}_1,\tilde{x}_2,\tilde{p}_2\}$ from the commutative variables $\{x_1,p_1,x_2,p_2\}$ by means of the following linear transformations:

$$\tilde{x}_i=x_i-\frac{1}{2}\theta_{ij}p_j, \quad \tilde{p}_i=p_i+\frac{1}{2}\eta_{ij}x_j \ . \tag{17}$$

The substitution of Eq. (17) into (16) gives

$$\begin{aligned} H \ = \ & \frac{1}{2m}(p_1^2+p_2^2+\eta x_2p_1-\eta x_1p_2)+ \\ & \frac{1}{2}k(x_1^2+x_2^2+\theta x_2p_1-\theta x_1p_2) \ , \end{aligned} \tag{18}$$

where the commutative variables $\{x_1,p_1,x_2,p_2\}$ satisfy the usual commutation relations:

$$\{x_i,x_j\}=0, \quad \{x_i,p_j\}=\delta_{ij}, \quad \{p_i,p_j\}=0. \tag{19}$$

The equations of motion of the harmonic oscillator are given by

$$\dot{x}_1=\{x_1,H\}=\frac{p_1}{m}+\frac{1}{2m}\eta x_2+\frac{1}{2}k\theta x_2 \ , \tag{20}$$

$$\dot{p}_1=\{p_1,H\}=-kx_1+\frac{1}{2}\eta\dot{x}_2+\frac{1}{2}k\theta m\dot{x}_2 \ . \tag{21}$$

This can be written as

$$m\ddot{x}_1=-kx_1+\eta\dot{x}_2+\theta km\dot{x}_2 \ . \tag{22}$$

and similar

$$m\ddot{x}_2=-kx_2-\eta\dot{x}_1-\theta km\dot{x}_1 \ . \tag{23}$$

Eq. (22) can also be obtained from Eq. (15) by taking $i=1$ and Eq. (23) by taking $i=2$. It is obvious that the equation of motion obtained by the second method is consistent with the one obtained by Newton's modified second law Eq. (11). Thus we have show that the so redefined Poisson brackets and new corrections to Newton's second law are correct in non-commutative phase space.

5 Conclusion

We have investigated the laws of motion of classical particles in a non-commutative phase space. The corresponding non-commutative relations contain not only the spatial non-commutativity but also the momentum non-commutativity. First, new Poisson brackets (Eq. (7)) have been defined in the non-commutative phase space and shown to contain corrections due to the non-commutativity of coordinates and momentum components. On the basis of this new Poisson brackets a new modification of Newton's second law (Eq. (11)) has been derived. Based on this the equations of motion (Eq. (13)) for a free particle have been derived. Eq. (13) shows that the acceleration of a free particle in commutative space is non-zero in non-commutative phase space. The term on the right hand side of Eq. (13) is an analogue of a damping force, caused by the non- commutativity of the momentum. In other words, a free particle in commutative space is not a free particle with zero-acceleration in non-commutative phase space. The reason for the non-vanishing acceleration is the non-commutativity of momentum. It remains, however, a free particle with zero-acceleration in non-commutative space if only the coordinates are non-commutative. In the second example, the harmonic oscillator, also a damping force appears (Eq. (15)), again caused by the non- commutativity of momenta and coordinates, however also depending on the presence of the external field. Finally we applied the linear transformation method[15] to solve the equation of motion for the harmonic oscillator. Both methods led to the same equations of motion, proving the correctness of the modified second law of Newton, obtained from the modified Poisson brackets.

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