Klein-Gordon oscillators in noncommutative phase space*

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Abstract We study the Klein-Gordon oscillators in non-commutative (NC) phase space. We find that the Klein-Gordon oscillators in NC space and NC phase-space have a similar behaviour to the dynamics of a particle in commutative space moving in a uniform magnetic field. By solving the Klein-Gordon equation in NC phase space, we obtain the energy levels of the Klein-Gordon oscillators, where the additional terms related to the space-space and momentum-momentum non-commutativity are given explicitly.

Key words noncommutative phase space, Landau problem, Klein-Gordon oscillators

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1 Introduction

There are many papers devoted to the study of various aspects of quantum mechanics in NC space and NC phase space with the usual time coordinate[1—15]. For example, the Aharonov-Bohm phase in NC space and NC phase space has been studied in Refs. [1—3]. The Aharonov-Casher phase for a spin-1/2 and spin-1 particle in NC space and NC phase space has been studied in Refs. [4—8]. The Landau problem in NC quantum mechanics has been discussed in Refs. [9—12]. Ref. [13] studied the Klein-Gordon oscillators in non-commutative space. It is still interesting to study the Klein-Gordon oscillators in non-commutative phase space.

This paper is organized as follows: in Section 2, we discuss the Klein-Gordon oscillators in NC space. In Section 3, we study the Klein-Gordon oscillators in NC phase space. In Section 4, by solving the Klein-Gordon equation, we deduce the energy levels of a particle in a magnetic field in NC phase space. A summary is given in the last section.

2 The Klein-Gordon oscillators in NC space

In NC space the coordinate $\hat{x}_i$ and momentum $\hat{p}_i$ operators satisfy the following commutation relations

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\theta_{ij},\quad [\hat{p}_i,\hat{p}_j]=0,\quad [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij}. \tag{1}$$

By replacing the normal product with a star product, the Schrödinger equation in commuting space will change into the Schrödinger equation in NC space.

$$H(p,x)*\psi(x)=E\psi(x), \tag{2}$$

where the Moyal-Weyl (or star) product between two functions is defined as

$$(f*g)(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}}f(x_i)g(x_j)= f(x)g(x)+\frac{\mathrm{i}}{2}\Theta_{ij}\partial_i f\partial_j g\big|_{x_i=x_j}+\mathcal{O}(\theta^2). \tag{3}$$

Here $f(x)$ and $g(x)$ are two arbitrary functions. Instead of solving the NC Schrödinger equation by using the star product procedure, we use Bopp's shift

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method, that is, we replace the star product in the Schrödinger equation by the usual product by making a Bopp's shift

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij}p_j, \quad \hat{p}_i = p_i. \tag{4}$$

Then the noncommutative Schrödinger equation can be solved in the commuting space, and the noncommutative properties can be realized by the θ related terms.

Studies in Refs.[9—12] have shown that the nonrelativistic harmonic oscillators in noncommutative space have properties also encountered in the Landau problem in commutative space. Now , following Ref. [13], we review the Klein-Gordon oscillators in NC space. The the Klein-Gordon oscillators in two dimensional commutative space is defined by the following equation

$$c^2(\boldsymbol{p}+\mathrm{i}mw\boldsymbol{r})\cdot(\boldsymbol{p}-\mathrm{i}mw\boldsymbol{r})\psi = (E^2-m^2c^4)\psi, \tag{5}$$

By a straightforward calculation (in 2 dimensions) we arrive at the following equation

$$c^2[(p_x^2+p_y^2)+m^2w^2(x^2+y^2)]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi, \tag{6}$$

with energy eigenvalues

$$E^2_{n_xn_y} = 2mc^2\hbar w(n_x+n_y+1)+m^2c^4-2mc^2\hbar w. \tag{7}$$

In a noncommutative space one may describe the Klein-Gordon oscillators by the following equation

$$c^2[(\boldsymbol{p}+\mathrm{i}mw\boldsymbol{r})\cdot(\boldsymbol{p}-\mathrm{i}mw\boldsymbol{r})]*\psi = (E^2-m^2c^4)\psi, \tag{8}$$

Instead of solving the NC Klein-Gordon Eq. (8) by using the star product, an equivalent method will be used in this paper, i.e., we replace the star product in the Klein-Gordon equation

$$c^2[(\hat{p}_x^2+\hat{p}_y^2)+m^2w^2(\hat{x}^2+\hat{y}^2)]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi, \tag{9}$$

by the usual product with a Bopp's shift Eq. (4). In the two dimensional non-commutative space, Eq. (4) becomes

$$\hat{x} = x - \frac{1}{2\hbar}\theta p_y, \ \ \hat{y} = y + \frac{1}{2\hbar}\theta p_x, \ \ \hat{p_x} = p_x, \ \ \hat{p_y} = p_y. \tag{10}$$

Inserting Eq. (10)into Eq. (9), we have

$$c^2\left[(p_x^2+p_y^2)+m^2w^2\left(x-\frac{1}{2\hbar}\theta p_y\right)^2 + m^2w^2\left(y+\frac{1}{2\hbar}\theta p_x\right)^2\right]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi. \tag{11}$$

By a straightforward calculation, we arrive at the following equation

$$c^2\left[\left(1+\frac{m^2w^2\theta^2}{4\hbar^2}\right)(p_x^2+p_y^2)+m^2w^2(x^2+y^2) - \frac{m^2w^2\theta}{\hbar}L_z\right]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi. \tag{12}$$

Neglecting terms with θ^2, we have

$$c^2\left[(p_x^2+p_y^2)+m^2w^2(x^2+y^2)-\frac{m^2w^2\theta}{\hbar}L_z\right]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi. \tag{13}$$

The energy eigenvalues are given by

$$E^2_{n_xn_ym_\ell} = 2mc^2\hbar\omega(n_x+n_y+1) - \left(\frac{m^2w^2c^2\theta}{\hbar}\right)m_\ell\hbar+m^2c^4-2mc^2\hbar w \tag{14}$$

and indicate a similarity to the normal Zeeman effect.

3 The Klein-Gordon oscillators in NC phase space

The Bose-Einstein statistics in non-commutative quantum mechanics requires both space-space and momentum-momentum non-commutativity. On NC phase space, we replace the commutation relations (1) by

$$[\hat{x}_i,\hat{x}_j] = \mathrm{i}\theta_{ij}, \quad [\hat{p}_i,\hat{p}_j] = \mathrm{i}\bar{\theta}_{ij}, \quad [\hat{x}_i,\hat{p}_j] = \mathrm{i}\hbar\delta_{ij}. \tag{15}$$

The Schrödinger equation in NC phase space is the same as given in Eq. (2), but the star product in Eq. (2), for NC phase space, is defined by,

$$(f*g)(x,p) = \mathrm{e}^{\frac{\mathrm{i}}{2\alpha^2}\theta_{ij}\partial_i^x\partial_j^x+\frac{\mathrm{i}}{2\alpha^2}\bar{\theta}_{ij}\partial_i^p\partial_j^p}f(x,p)g(x,p) = f(x,p)g(x,p)+\frac{\mathrm{i}}{2\alpha^2}\theta_{ij}\,\partial_i^x f\,\partial_j^x g\big|_{x_i=x_j} + \frac{\mathrm{i}}{2\alpha^2}\bar{\theta}_{ij}\,\partial_i^p f\,\partial_j^p g\big|_{p_i=p_j}+\mathcal{O}(\theta^2), \tag{16}$$

where $\mathcal{O}(\theta^2)$ stands for the second and higher order terms of θ and $\bar{\theta}$. In NC phase space the star product in the Schrödinger equation can be replaced by a generalized Bopp's shift, i.e., the non-commutative coordinates and momenta are shifted by

$$x_i \to \hat{x}_i = \alpha x_i - \frac{1}{2\alpha\hbar}\theta_{ij}p_j, \quad p_i \to \hat{p}_i = \alpha p_i + \frac{1}{2\alpha\hbar}\bar{\theta}_{ij}x_j. \tag{17}$$

Now we are in the position to discuss the energy levels of the Klein-Gordon oscillators in NC phase space . In two dimensional NC phase space, Eq. (17) becomes

$$\hat{x} = \alpha x - \frac{1}{2\alpha\hbar}\theta p_y, \ \ \hat{y} = \alpha y + \frac{1}{2\alpha\hbar}\theta p_x, \quad \hat{p_x} = \alpha p_x + \frac{1}{2\alpha\hbar}\bar{\theta}y, \ \ \hat{p_y} = \alpha p_y - \frac{1}{2\alpha\hbar}\bar{\theta}x. \tag{18}$$

Inserting Eq. (18) into Eq. (9), we have

$$c^2\left\{\left(\alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y\right)^2+\left(\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\right)^2+ m^2w^2\left[\left(\alpha x-\frac{1}{2\alpha\hbar}\theta p_y\right)^2+\left(\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\right)^2\right]\right\}\psi= (E^2-m^2c^4+2mc^2\hbar w)\psi. \tag{19}$$

With a similar procedure as in NC space, we obtain the following Klein-Gordon equation in NC phase space

$$c^2\left[\alpha^2(p_x^2+p_y^2)+\alpha^2m^2w^2(x^2+y^2)-\frac{\bar{\theta}+m^2w^2\theta}{\hbar}L_z\right]\psi= (E^2-m^2c^4+2mc^2\hbar w)\psi, \tag{20}$$

and the energy eigenvalues are given by

$$E^2_{n_xn_ym_\ell}=2mc^2\hbar\Omega(n_x+n_y+1)- \left(\frac{c^2\bar{\theta}+m^2w^2c^2\theta}{\hbar}\right)m_\ell\hbar+m^2c^4-2mc^2\hbar w, \tag{21}$$

where

$$\Omega=w\alpha^2\,. \tag{22}$$

The energy levels $E^2_{n_xn_ym_\ell}$ represent both, space-space and momentum-momentum non-commutativity. In a 2 dimensional non-commutative plane, $\bar{\theta}_{ij}=\bar{\theta}\epsilon_{ij}$, and the two NC parameters θ and $\bar{\theta}$ are related by $\bar{\theta}=4\alpha^2\hbar^2(1-\alpha^2)/\theta$[15]. If $\alpha=1$, then $\bar{\theta}_{ij}=0$, and the $E^2_{n_xn_ym_\ell}$ (Eq. (21)) in NC phase space will return to $E^2_{n_xn_ym_\ell}$ (Eq. (13)) in NC space.

By comparing Eq. (12) and Eq. (20) with the Landau problem in non-relativistic quantum mechanics, one finds that the Klein-Gordon oscillators in non-commutative space and noncommutative phase-space have similar properties as the dynamics of a particle in a uniform magnetic field in a commutative space.

4 Energy levels of the Klein-Gordon equation for a particle in a uniform magnetic field in NC phase space

In this section we discuss the energy levels of the Klein-Gordon equation for a particle in a uniform magnetic field in NC phase space. The Klein-Gordon equation for a particle in a uniform magnetic field in a commutative space can be written as

$$c^2\left[\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)\right]\cdot\left[\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)\right]\psi=(E^2-m^2c^4)\psi, \tag{23}$$

where

$$\boldsymbol{A}=\frac{\boldsymbol{B}\times\boldsymbol{r}}{2}. \tag{24}$$

Substituting Eq. (24) into Eq. (23)one gets

$$c^2[(p_x^2+p_y^2)+\left(\frac{e^2B^2}{4c^2}\right)(x^2+y^2)- \frac{eB}{c}(xp_y-yp_x)]\psi=(E^2-m^2c^4)\psi. \tag{25}$$

By comparing Eq. (25) with the Eq. (12) and Eq. (20) one finds that, even in the relativistic case, the Klein-Gordon oscillators in non-commutative space and noncommutative phase-space also have a behaviour similar to the dynamics of a particle in a uniform magnetic field in commutative space.

In NC phase space Eq. (25) can be written as

$$c^2\left[(p_x^2+p_y^2)+\left(\frac{e^2B^2}{4c^2}\right)(x^2+y^2)- \frac{eB}{c}(xp_y-yp_x)\right]*\psi=(E^2-m^2c^4)*\psi. \tag{26}$$

After replacing the star product with the shift defined in Eq. (18), one obtains

$$c^2\left\{\left(\alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y\right)^2+\left(\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\right)^2+ m^2\omega_1^2\left[\left(\alpha x-\frac{1}{2\alpha\hbar}\theta p_y\right)^2+\left(\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\right)^2\right]- 2m\omega_1\left[\left(\alpha x-\frac{1}{2\alpha\hbar}\theta p_y\right)\left(\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\right)- \left(\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\right)\left(\alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y\right)\right]\right\}\psi= (E^2-m^2c^4)\psi, \tag{27}$$

where

$$\omega_1=\frac{eB}{2mc}\,. \tag{28}$$

By a further simplification we get the Klein-Gordon equation for a particle in a constant magnetic field in NC phase space as

$$c^2\left[\left(\alpha^2+\frac{m\omega_1\theta}{\hbar}\right)(p_x^2+p_y^2)+ \left(\alpha^2+\frac{\bar{\theta}}{\hbar m\omega_1}\right)m^2\omega_1^2(x^2+y^2)- \frac{\bar{\theta}+m^2\omega_1^2\theta+2m\hbar\omega_1}{\hbar}L_z\right]\psi=(E^2-m^2c^4)\psi. \tag{29}$$

The energy eigenvalues are given by

$$E^2_{n_xn_ym_\ell}=2mc^2\hbar\Omega_1(n_x+n_y+1)- \left(\frac{c^2\bar{\theta}+m^2\omega_1^2c^2\theta+2mc^2\hbar\omega_1}{\hbar}\right)m_\ell\hbar+m^2c^4, \tag{30}$$

where

$$\Omega_1 = \omega_1 \sqrt{\alpha^2 + \frac{m\omega_1\theta}{\hbar}} \sqrt{\alpha^2 + \frac{\bar{\theta}}{\hbar m \omega_1}} \ . \qquad (31)$$

The energy levels $E^2_{n_x n_y m_\ell}$ contain the effects of both, space-space and momentum-momentum non-commutativity.

5 Summary

First, we discussed the energy levels of the Klein-Gordon oscillators in NC space. Then we obtained the energy levels of the Klein-Gordon oscillators in NC phase space. At last, we obtained the energy levels of the Klein-Gordon oscillators for a particle in a constant magnetic field in NC phase space. We note that the known similarity between an oscillators in non-commutative space and a particle in a constant magnetic field[9—12] can be extended to relativistic motion.

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