Improved Faddeev-Jackiw quantization of the electromagnetic field and Lagrange multiplier fields*

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Abstract We use the improved Faddeev-Jackiw quantization method to quantize the electromagnetic field and its Lagrange multiplier fields. The method's comparison with the usual Faddeev-Jackiw method and the Dirac method is given. We show that this method is equivalent to the Dirac method and also retains all the merits of the usual Faddeev-Jackiw method. Moreover, it is simpler than the usual one if one needs to obtain new secondary constraints. Therefore, the improved Faddeev-Jackiw method is essential. Meanwhile, we find the new meaning of the Lagrange multipliers and explain the Faddeev-Jackiw generalized brackets concerning the Lagrange multipliers.

Key words Faddeev-Jakiw method, constraint, quantization, electromagnetic field, Lagrange multiplier

PACS 11.10.-z

1 Introduction

Systems described by singular Lagrangians are called singular systems and this kind of systems contains inherent constraints[1, 2]. The electromagnetic field theory[1, 2] and Yang-Mills theory[1, 2] are singular systems. In many domains of physics, there exist different singular systems, such as the gauge field theories, the gravitational field theory, the supersymmetric theory, supergravity, the superstring theory and so on. The investigation on inherent constraints has become one basic task of theoretical research in these theories.

The study of singular systems was started by Dirac[3], who proposed a kind of bracket (Dirac bracket) to quantize singular systems. However, the Faddeev-Jackiw method is another fundamental quantization method showing up in the 80's of the 20th century. In contrast to the Dirac method, it has some very useful properties of obviating the need to distinguish primary & secondary constraints and first & second types of constraints. The method is simpler and does not rely on a hypothesis such as Dirac's conjecture. Thus it has evoked much attention. In the development of the Faddeev-Jackiw method, the authors of Ref. [4] studied the method and proposed a new kind of brackets. Subsequently in Ref. [5] its reasonableness was demonstrated and the Faddeev-Jackiw method systematically developed. In succession, Refs. [6—8] presented the procedure of dealing with constraints in the Faddeev-Jackiw method, and Ref. [9] further gave the Faddeev-Jackiw quantization method of path integrals.

Refs. [10, 11] proved the equivalence of the Dirac method and the Faddeev-Jackiw method for systems with no constraints, and in Ref. [12] the equivalence between the original Faddeev-Jackiw method[5] and the Dirac method in such systems has been discussed. However, in Ref. [13] it was proved that the usual Faddeev-Jackiw method[6—8] and the Dirac method were not completely equivalent. In particular for Lagrangians with in which by assumption no variables are eliminated in the Faddeev-Jackiw formalism, the constraints calculated by the two methods are not consistent. In Ref. [13] it was shown that some constraints, when calculated in Dirac formalism, don't appear in the calculation in the Faddeev-Jackiw formalism. This results in a contradiction between the

Received 26 November 2007, Revised 4 April 2008
* Supported by Beijing Natural Science Foundation (1072005)

usual Faddeev-Jackiw quantization and the Dirac quantization.

Hence, in this paper we use the improved Faddeev-Jackiw method[13] to quantize the electromagnetic field and its Lagrange multiplier fields, and further prove that the improved method proposed in Ref. [13] is an economic and effective quantization method for practical applications.

This paper is organized as follows: Sect. 2 gives the improved Faddeev-Jackiw quantization of the electromagnetic field and the Lagrange multiplier fields, and a comparison with the usual Faddeev-Jackiw and Dirac method; in Sect. 3 the new interpretation of the Lagrange multipliers is given; and the last section provides a summary and the conclusion.

2 Improved Faddeev-Jackiw quantization of electromagnetic fields and Lagrange multiplier fields

The Lagrangian density of the electromagnetic field is given by

$$L=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $g_{\mu\nu}=g^{\mu\nu}=\mathrm{diag}\{1,-1,-1,-1\}$. Eq. (1) is not a first-order Lagrangian density. So before performing the improved Faddeev-Jackiw quantization process, it must be transformed into a first-order Lagrangian density by introducing auxiliary fields. Therefore, we can choose the canonical momenta as auxiliary fields.

The canonical momenta are defined as follows

$$\pi^\mu=\frac{\partial L}{\partial \dot{A}_\mu}=-F^{0\mu}\ . \tag{2}$$

Correspondingly, we have

$$\pi^\mu\dot{A}_\mu-L=\frac{1}{2}\pi_i\pi_i+A_0\,\partial_i\,\pi_i+\frac{1}{4}F_{ij}F^{ij}\ , \tag{3}$$

so the first order symplectic Lagrangian density is

$$L^{(0)}=-\pi_i\dot{A}_i-V^{(0)}\ , \tag{4}$$

where the divergence term $\partial_i(A_0\pi_i)$ of the Lagrangian density is neglected because of $A_0\pi_i=0$ at infinity in general field theory, and the symplectic potential $V^{(0)}$ is given by

$$V^{(0)}=\frac{1}{2}\pi_i\pi_i+A_0\,\partial_i\,\pi_i+\frac{1}{4}F_{ij}F^{ij}\ . \tag{5}$$

The corresponding symplectic equations of motion are

$$f_{ij}^{(0)}\dot{\xi}^j=\frac{\partial V^{(0)}(\xi)}{\partial\xi^i}\ , \tag{6}$$

where

$$f_{ij}^{(0)}(x,y)=\frac{\delta a_j(y)}{\delta\xi^i(x)}-\frac{\delta a_i(x)}{\delta\xi^j(y)}.$$

We take the set of symplectic variables to be

$$\xi^{(0)}(x)=\{A_i,\,\pi_i,\,A_0\}. \tag{7}$$

The components of the symplectic 1-forms are

$$a_{A_i}^{(0)}=-\pi_i,\quad a_{\pi_i}^{(0)}=0,\quad a_{A_0}^{(0)}=0. \tag{8}$$

Thus we obtain the symplectic matrix as

$$f_{ij}^{(0)}(x,y)=\begin{pmatrix}0 & \delta_{ij} & 0\\ -\delta_{ij} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\cdot\delta(x-y), \tag{9}$$

which is obviously singular. The zero-mode of this matrix is $(v^{(0)})^{\mathrm{T}}=(0,0,v^{A_0})$, where v^{A_0} is an arbitrary function. In terms of the Faddeev-Jackiw method[14], using the zero-mode we can get the primary constraints as

$$\varOmega^{(0)}=(v^{(0)})_i^{\mathrm{T}}\frac{\partial V^{(0)}(\xi)}{\partial\xi^i}=v^{A_0}\cdot\partial_i\,\pi_i=0. \tag{10}$$

So far, it is not different from the usual Faddeev-Jackiw method, but the difference will appear in the following. The reason for the inconsistency of the usual Faddeev-Jackiw method and the Dirac method is, that in the usual Faddeev-Jackiw method with the assumption of no elimination of variables, one introduces Lagrangian multipliers for the constraints to construct a new Lagrangian. Usually, people consider the equations of motion induced by this new Lagrangian as the real equations of motion and use these equations to calculate new constraints. However, the real equations of motion are only those induced by the initial Lagrangian[13] (i.e., Eq. (4)). Moreover, all Faddeev-Jackiw constraints are inherent constraints. So, introducing Lagrangian multipliers for them is not allowed[13].

Therefore, we use the improved Faddeev-Jackiw method to avoid such an inconsistency and do not introduce Lagrangian multipliers for Eq. (10) into the Lagrangian to construct a new one, but use the consistency condition analogous to the Dirac-Bergmann algorithm[15, 16]

$$\dot{\varOmega}^{(0)}=\frac{\partial\,\varOmega^{(0)}}{\partial\xi^i}\dot{\xi}^i=0 \tag{11}$$

to deduce new constraints.

On the other hand, because general physical processes should satisfy quantitative causal relations with no-loss-no-gain character[17, 18], e.g., Ref. [19] uses the no-loss-no-gain homeomorphic map transformation satisfying a quantitative causal relation to gain exact strain tensor formulas in a Weitzenböck manifold. In fact, some changes (cause) of some quantities in Eq. (11) must result in some relative changes (result) of the other quantities in Eq. (11) so that Eq. (11)'s right side keeps no-loss-no-gain, i.e., it remains zero. This means that Eq. (11) also satisfies a quantitative causal relation, which enables the different quantities to form a useful expression.

Combining Eq. (11) with Eq. (6), we obtain the linear equations

$$\begin{cases} f_{ij}^{(0)}\dot{\xi}^j = \dfrac{\partial V^{(0)}(\xi)}{\partial \xi^i} \\ \dfrac{\partial \Omega^{(0)}}{\partial \xi^i}\dot{\xi}^i = 0 \end{cases} . \tag{12}$$

We can reformulate Eq. (12) as

$$f_{kj}^{(1)}\dot{\xi}^j = Z_k(\xi) \ , \tag{13}$$

where

$$f_{kj}^{(1)} = \begin{pmatrix} f_{ij}^{(0)} \\ \dfrac{\partial \Omega^{(0)}}{\partial \xi^i} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \partial_j & 0 \end{pmatrix} \cdot \delta(x-y)$$

and

$$Z_k(\xi) = \begin{pmatrix} \dfrac{\partial V^{(0)}(\xi)}{\partial \xi^i} \\ 0 \end{pmatrix}.$$

The coefficient matrix $(f_{kj}^{(1)})$ of Eq. (13) is obviously not a square matrix, but it still has a linearly independent mode $(v^{(1)})_k^{\mathrm{T}} = (\partial_i v^{\lambda}, 0, v^{A_0}, -v^{\lambda})$. Multiplication of $(f_{kj}^{(1)})$ by $(v^{(1)})_k^{\mathrm{T}}$ from the left side gives zero. Multiplying this mode to the two sides of Eq. (13), leads to the constraints

$$(v^{(1)})_k^{\mathrm{T}} Z_k = 0. \tag{14}$$

Substituting $\Omega^{(0)} = 0$ into Eq. (14), gives

$$(v^{(1)})_k^{\mathrm{T}} Z_k \Big|_{\Omega^{(0)}=0} = 0 \ . \tag{15}$$

Such a substitution guarantees that the obtained constraints do not appear in the following calculation.

If in the general case Eq. (15) is an identity, there is no new constraint. If, however, Eq. (15) is no identity, we have

$$\Omega^{(1)} = (v^{(1)})_k^{\mathrm{T}} Z_k \Big|_{\Omega^{(0)}=0} = 0 \ , \tag{16}$$

which is a secondary constraint. Similarly, introducing the consistency condition

$$\dot{\Omega}^{(1)} = \frac{\partial \Omega^{(1)}}{\partial \xi^i}\dot{\xi}^i = 0, \tag{17}$$

we can combine Eq. (12) with Eq. (17) to construct a group of new linear equations. With the help of these linear equations one detects step by step whether there are more new constraints, until there are no new constraints and we get the identity.

However, in this model, we can prove that Eq. (15) is an identity, so there is no new constraint, and the procedure using consistency conditions to obtain new constraints finishes. Then, in the next step, we can use the corresponding procedure of the usual Faddeev-Jackiw method. Therefore, comparing with the usual Faddeev-Jackiw method, the two methods are equivalent for their quantization results in this system. However, the process of getting new constraints in the improved method is simpler and more effective than the usual one.

Similar to the usual Faddeev-Jackiw method, we now introduce Lagrangian multiplier λ with respect to Eq. (10) into the Lagrangian to construct a new one

$$L^{(1)} = -\pi_i \dot{A}_i + (\partial_i \pi_i)\dot{\lambda} - V^{(1)}, \tag{18}$$

where $V^{(1)} = V^{(0)}\big|_{\partial_i \pi_i = 0} = \dfrac{1}{2}\pi_i\pi_i + \dfrac{1}{4}F_{ij}F^{ij}$. We also consider λ as a symplectic variable, and take a 1$^{\mathrm{st}}$-order symplectic variable set

$$\xi^{(1)}(x) = \{A_i,\, \pi_i,\, \lambda\}. \tag{19}$$

Then the corresponding components of the symplectic 1-form are given by

$$a_{A_i}^{(1)} = -\pi_i,\ a_{\pi_i}^{(1)} = 0,\ a_{\lambda}^{(1)} = \partial_i \pi_i \ . \tag{20}$$

The symplectic matrix is deduced as follows

$$f_{ij}^{(1)}(x,y) = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & \partial_i \\ 0 & -\partial_j & 0 \end{pmatrix} \cdot \delta(x-y), \tag{21}$$

which is still singular. Therefore this system has a gauge symmetry. Here, we choose the gauge condition $\Omega = \partial_i A_i = 0$, for which we introduce a Lagrange multiplier η to construct a new Lagrangian

$$L^{(2)} = -\pi_i \dot{A}_i + (\partial_i \pi_i)\dot{\lambda} + (\partial_i A_i)\dot{\eta} - V^{(2)}, \tag{22}$$

where

$$V^{(2)} = V^{(1)}\big|_{\partial_i A_i = 0} = \frac{1}{2}\pi_i\pi_i - \frac{1}{2}A_i \partial_j \partial_j A_i.$$

The 2nd-order symplectic variable set is $\xi^{(2)}(x) = \{A_i, \pi_i, \lambda, \eta\}$ and the corresponding components of the symplectic 1-form are

$$a_{A_i}^{(2)} = -\pi_i,\ a_{\pi_i}^{(2)} = 0,\ a_{\lambda}^{(2)} = \partial_i \pi_i,\ a_{\eta}^{(2)} = \partial_i A_i \ . \tag{23}$$

Finally, we obtain the new symplectic matrix as

$$f_{ij}^{(2)}(x,y) = \begin{pmatrix} 0 & \delta_{ij} & 0 & \partial_i \\ -\delta_{ij} & 0 & \partial_i & 0 \\ 0 & -\partial_j & 0 & 0 \\ -\partial_j & 0 & 0 & 0 \end{pmatrix} \cdot \delta(x-y), \tag{24}$$

which is a non-singula, reversible matrix. Consequently, its inverse is given by

$$(f_{ij}^{(2)}(x,y))^{-1} = \begin{pmatrix} 0 & -\delta_{ij}+\frac{\partial_i\partial_j}{\nabla^2} & 0 & -\frac{\partial_j}{\nabla^2} \\ \delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2} & 0 & -\frac{\partial_j}{\nabla^2} & 0 \\ 0 & \frac{\partial_i}{\nabla^2} & 0 & -\frac{1}{\nabla^2} \\ \frac{\partial_i}{\nabla^2} & 0 & \frac{1}{\nabla^2} & 0 \end{pmatrix} \cdot\delta(x-y), \tag{25}$$

from which, we can identify the Faddeev-Jackiw generalized brackets[7] as

$$\left\{\xi_i^{(2)}(x),\xi_j^{(2)}(y)\right\}_* = (f_{ij}^{(2)}(x,y))^{-1}. \tag{26}$$

The quantization of the electromagnetic field is done by the usual replacement

$$\left\{\xi_i^{(2)}(x),\xi_j^{(2)}(y)\right\}_* \to -\frac{\mathrm{i}}{\hbar}\left[\hat{\xi}_i^{(2)}(x),\hat{\xi}_j^{(2)}(y)\right]. \tag{27}$$

Thus,

$$\left\{A_i(x),\pi_j(y)\right\}_* = (f_{12}^{(2)}(x,y))^{-1} = \left(-\delta_{ij}+\frac{\partial_i\partial_j}{\nabla^2}\right)\delta(x-y), \tag{28}$$

$$\left\{A_i(x),A_j(y)\right\}_* = (f_{11}^{(2)}(x,y))^{-1} = 0, \tag{29}$$

$$\left\{\pi_i(x),\pi_j(y)\right\}_* = (f_{22}^{(2)}(x,y))^{-1} = 0, \tag{30}$$

$$\left\{\lambda(x),\pi_i(y)\right\}_* = (f_{32}^{(2)}(x,y))^{-1} = \frac{\partial_i}{\nabla^2}\delta(x-y), \tag{31}$$

$$\left\{\eta(x),A_i(y)\right\}_* = (f_{41}^{(2)}(x,y))^{-1} = \frac{\partial_i}{\nabla^2}\delta(x-y), \tag{32}$$

$$\left\{\lambda(x),\eta(y)\right\}_* = (f_{34}^{(2)}(x,y))^{-1} = -\frac{1}{\nabla^2}\delta(x-y). \tag{33}$$

The other Faddeev-Jackiw generalized brackets are equal to zero. So far, we have completed the improved Faddeev-Jackiw quantization of this system.

Comparing Eqs. (28)—(30) with the quantization results of the Dirac method, we obtain

$$\left\{A_i(x),\pi_j(y)\right\}_* = (f_{12}^{(2)}(x,y))^{-1} = \left(-\delta_{ij}+\frac{\partial_i\partial_j}{\nabla^2}\right)\delta(x-y) = \left\{A_i(x),\pi_j(y)\right\}_{\mathrm{D}}, \tag{34}$$

$$\left\{A_i(x),A_j(y)\right\}_* = (f_{11}^{(2)}(x,y))^{-1} = 0 = \left\{A_i(x),A_j(y)\right\}_{\mathrm{D}}, \tag{35}$$

$$\left\{\pi_i(x),\pi_j(y)\right\}_* = (f_{22}^{(2)}(x,y))^{-1} = 0 = \left\{\pi_i(x),\pi_j(y)\right\}_{\mathrm{D}}. \tag{36}$$

At the same time, we emphasize that the Faddeev-Jackiw generalized brackets concerning λ and η are due to the Lagrangian multipliers, so there is no correspondence to the Dirac brackets. Comparing the Faddeev-Jackiw generalized brackets Eqs. (28)—(30) with Dirac brackets[1,2] for the real physical field variables, one finds that the two kinds of brackets are equivalent.

3 New interpretation of the Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$

In some pioneer works, such as Refs. [4—7], the Lagrange multipliers are explained as auxiliary fields that are introduced in order to give an extended symplectic tensor. Meanwhile, the brackets involving Lagrange multipliers do not appear in the Dirac method, in a sense, the brackets are not strong relations for the constraints. We now explain the new meanings of the Lagrange multipliers.

Using Eq. (22), we obtain the corresponding symplectic equations of motion

$$f_{ij}^{(2)}\dot{\xi}^{(2)j} = \frac{\partial V^{(2)}(\xi)}{\partial \xi^{(2)i}}, \tag{37}$$

or in components

$$\begin{pmatrix} 0 & \delta_{ij} & 0 & \partial_i \\ -\delta_{ij} & 0 & \partial_i & 0 \\ 0 & -\partial_j & 0 & 0 \\ -\partial_j & 0 & 0 & 0 \end{pmatrix}\cdot\delta(x-y)\cdot\begin{pmatrix} \dot{A}_j \\ \dot{\pi}_j \\ \dot{\lambda} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -\nabla^2 A_i \\ \pi_i \\ 0 \\ 0 \end{pmatrix}\cdot\delta(x-y), \tag{38}$$

so we can have

$$\begin{cases} \dot{\pi}_j\delta_{ij}+\partial_i\dot{\eta} = -\nabla^2 A_i \\ -\dot{A}_j\delta_{ij}+\partial_i\dot{\lambda} = \pi_i \\ -\partial_j\dot{\pi}_j = 0 \\ -\partial_j\dot{A}_j = 0 \end{cases}. \tag{39}$$

The solutions of these equations are

$$\begin{cases} \lambda = \int \mathrm{d}x_i\mathrm{d}t(\pi_i+\dot{A}_i) \\ \eta = \int \mathrm{d}x_i\mathrm{d}t(-\dot{\pi}_i-\nabla^2 A_i) \end{cases}. \tag{40}$$

According to Eq. (40) the auxiliary fields λ and η are no real independent physical fields. Instead they are just some combinations or functions of the real physical fields A_i and π_i. We can also derive their Faddeev-Jackiw generalized brackets Eqs. (31)—(33).

From the above results we conclude that the quantization commutation relations of the Faddev-Jackiw brackets for the Lagrange multipliers are just the commutation relations of the combinations or functions of the real physical fields A_i and π_i.

4 Summary and conclusion

We have quantized the electromagnetic field and its Lagrange multiplier fields by the improved Faddeev-Jackiw quantization method. A comparison with the usual Faddeev-Jackiw method and Dirac method is also given. We find that the improved Faddeev-Jackiw method is equivalent to the usual Faddeev-Jackiw method and the Dirac method in quantizing this system. It was shown that the improved Faddeev-Jackiw method retains all the virtues of the usual Faddeev-Jackiw method, obviates the need to distinguish primary and secondary constraints and between the first and the second type of constraints. Thus, the method is simpler, and does not need such a hypothesis as Dirac's conjecture.

Furthermore, the improved Faddeev-Jackiw method is simpler than the usual one when obtaining new secondary constraints. Therefore, the improved Faddeev-Jackiw method is a more economical and effective method of canonical quantization. We also give a new interpretation of the Lagrange multipliers, namely, they are just some combinations or functions of the real physical fields in this system.

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