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3-to-3 Parton Elastic Scatterings and Early Thermalization*

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Abstract Triple-gluon elastic scatterings ggg $\rightarrow$ ggg and triple-quark elastic scatterings qqq $\rightarrow$ qqq are studied. The H-theorem extended to the transport equations with the $3\rightarrow3$ elastic scatterings is proved. A short thermalization time for gluon matter and a long thermalization time for quark matter are results of the transport equations.

Key words triple-gluon scattering, triple-quark scattering, transport equation, thermalization

1 Introduction

Quark-gluon plasma (QGP) is taken as a (locally) thermally equilibrated state of matter in which quarks and gluons are deconfined from hadrons, so that color degrees of freedom become manifest over nuclear, rather than merely nuleonic, volumes[1]. In order to assert that QGP is observed, whether or not a thermal state is established must be confirmed. Therefore, the study of thermalization is a very important and basic task in gold-gold nuclear collisions at the Relativistic Heavy Ion Collider (RHIC)[1, 2]. Because the deconfined matter has a limited life-time of a few fm/c[3], the thermalization must be completed before the deconfined matter fades. Fortunately, a thermalization time of the order less than 1fm/c is concluded from the hydrodynamic calculations of the low-p_T elliptic flow of hadrons observed at RHIC[4—9]. The rapid thermalization i.e. the early thermalization that takes place in a time less than 1fm/c is a new phenomenon that does support the existence of QGP. Then explaining the early thermalization from QCD is a very important task. The two-parton to two-parton scatterings and $2\rightarrow3$ scatterings give a thermalization time greater than 1fm/c for gluon matter[10, 11]. The 3-to-3 gluon elastic scatterings can reduce the thermalization time of gluon matter to be smaller than 1fm/c[12]. The occurrence of the 3-to-3 gluon elastic scatterings is the result of high gluon number density that is achieved in initial Au-Au collisions at RHIC energies. The early thermalization is an effect of the 3-to-3 gluon elastic scatterings. We can expect that the many-body color interactions play an important role in ultrarelativistic heavy-ion collisions[13].

In Au-Au collisions the two incoming beams contain partons moving along the beam directions. Hard and semihard scatterings produce partons which mostly distribute near the beam directions. Hence the distribution of partons is anisotropic in the initial Au-Au collisions. On the other hand, the produced partons are within a pancake volume since the two incoming nuclei are highly Lorentz-contracted along the beam directions, then the density of produced partons can be very high, for instance, 38fm^{-3}. Scatterings of such high density partons will convert the anisotropic distribution into an isotropic distribution. The 3-parton elastic scatterings are important in the convertion process since the ratio of the number of 3-parton scatterings to the number of 2-parton scatterings is around 0.7[14]. The counting of the number of three-parton scatterings or two-parton scatterings

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is based on an anisotropic momentum distribution obtaind from the HIJING simulation[15] for initial central Au-Au collisions at $\sqrt{s_{\rm NN}}=200$GeV. If the distance of two partons is smaller than a given interaction range, a scattering of the two partons occurs. If three partons are within a sphere whose center is the center of mass of the three partons and whose radius equals the given interaction range, a scattering of the three partons occurs. For a typical interaction range that is larger than 0.1fm, the ratio varies around 0.7. The value of the ratio convinces us of the importance of the three-parton scatterings at high parton number density.

Quark-gluon matter contains gloun matter, quark matter and antiquark matter. The thermalization of gluon matter and quark matter is individually studied for the understanding of the role of the three-gluon elastic scatterings or of the three-quark elastic scatterings. In the next section, the three-gluon elastic scatterings and the three-quark elastic scatterings are stated briefly. Transport equations and numerical results are given in Section 3 and the summary is given in the last section.

2 Three-gluon and three-quark elastic scatterings

The three-gluon scatterings ggg → ggg are very complicated processes. Diagrams of the three-gluon elastic scatterings and the three-quark elastic scatterings in the present work are at tree level that is of order $\alpha_{\rm s}^4$. Of all the diagrams six typical diagrams as shown in Figs. 1 and 2 are selected to illustrate the elastic scattering processes.

The process shown by the diagram $B_{\sim\sim}$ indicates that three initial gluons at a space-time point are annihilated by a four-gluon coupling into a virtual gluon, which will decay into three final real gluons at another space-time point. The squared amplitude of this diagram can be derived by hand and the complexity of the derivation originates from products of eight $SU(3)$ structure constants and products of the six polarization 4-vectors for the six gluons. The squared amplitude obtained in Appdendix A depends only on the center-of-mass energy of the three initial gluons, $\sqrt{s}$, in an inverse square form.

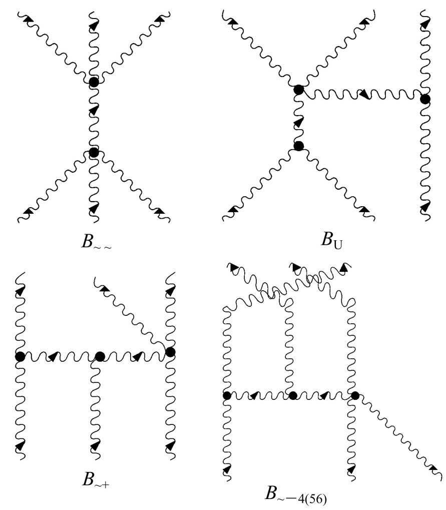

Fig. 1. The scatterings of three gluons.

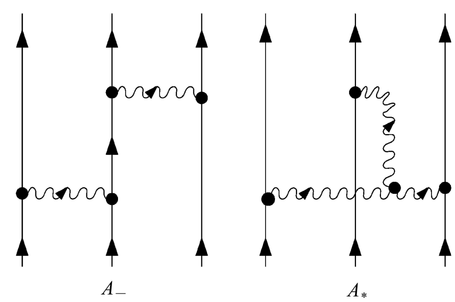

Fig. 2. The scatterings of three quarks.

The diagram $B_{\rm U}$ involves two initial gluons annihilating into a virtual gluon which decays into two final real gluons with radiating or absorbing a virtual gluon interacting with the other initial gluon. The diagram $B_{\sim+}$ indicates that two initial gluons scatter into a final real gluon and a virtual gluon which scatters with the other initial gluon into two final real gluons. The diagram $B_{\sim-4(56)}$ shows that two initial gluons scatter into two final real gluons and one virtual gluon which is absorbed by the other initial gluon to form a final real gluon, and the three final gluons exchange. The squared amplitudes of the three diagrams can not be derived by hand and instead have to be derived by Fortran codes.

In Diagram A_{-}, two on-shell initial quarks scatter into an on-shell quark and an off-shell quark which scatters further with the other on-shell initial quark to produce two on-shell quarks. In Diagram A_{*} three on-shell initial quarks scatter into three on-shell final quarks by the triple-gluon coupling.

3 Transport equations and results

The spin- and color-averaged squared amplitude of the 3-to-3 gluon elastic scatterings contributes to the variation of gluon distribution in gluon matter by the transport equation

$$\begin{aligned}\frac{\partial f_{g1}}{\partial t}+\boldsymbol{v}_1\cdot\nabla_{\boldsymbol{r}}f_{g1} = & -\frac{g_{\rm G}}{2E_1}\int\frac{{\rm d}^3p_2}{(2\pi)^32E_2}\frac{{\rm d}^3p_3}{(2\pi)^32E_3}\frac{{\rm d}^3p_4}{(2\pi)^32E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\\ & \frac{1}{2}|\mathscr{M}_{\rm gg\to gg}|^2[f_{g1}f_{g2}(1+f_{g3})(1+f_{g4})-f_{g3}f_{g4}(1+f_{g1})(1+f_{g2})]-\\ & \frac{g_{\rm G}^2}{2E_1}\int\frac{{\rm d}^3p_2}{(2\pi)^32E_2}\frac{{\rm d}^3p_3}{(2\pi)^32E_3}\frac{{\rm d}^3p_4}{(2\pi)^32E_4}\frac{{\rm d}^3p_5}{(2\pi)^32E_5}\frac{{\rm d}^3p_6}{(2\pi)^32E_6}\times\\ & (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\frac{1}{12}|\mathscr{M}_{\rm ggg\to ggg}|^2\times\\ & [f_{g1}f_{g2}f_{g3}(1+f_{g4})(1+f_{g5})(1+f_{g6})-f_{g4}f_{g5}f_{g6}(1+f_{g1})(1+f_{g2})(1+f_{g3})]\ . \end{aligned} \tag{1}$$

In quark matter with equal distributions of up quarks and down quarks, the 3-to-3 quark elastic scatterings with the same or different quark flavors contribute to the up-quark distribution variation by the transport equation

$$\begin{aligned}\frac{\partial f_{q1}}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}f_{q1} = & -\frac{g_{\rm Q}}{2E_1}\int\frac{{\rm d}^3p_2}{(2\pi)^32E_2}\frac{{\rm d}^3p_3}{(2\pi)^32E_3}\frac{{\rm d}^3p_4}{(2\pi)^32E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\\ & \left(\frac{1}{2}|\mathscr{M}_{\rm uu\to uu}|^2+|\mathscr{M}_{\rm ud\to ud}|^2\right)[f_{q1}f_{q2}(1-f_{q3})(1-f_{q4})-f_{q3}f_{q4}(1-f_{q1})(1-f_{q2})]-\\ & \frac{g_{\rm Q}^2}{2E_1}\int\frac{{\rm d}^3p_2}{(2\pi)^32E_2}\frac{{\rm d}^3p_3}{(2\pi)^32E_3}\frac{{\rm d}^3p_4}{(2\pi)^32E_4}\frac{{\rm d}^3p_5}{(2\pi)^32E_5}\frac{{\rm d}^3p_6}{(2\pi)^32E_6}\times\\ & (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\times\\ & \left[\frac{1}{12}|\mathcal{M}_{\rm uuu\to uuu}|^2+\frac{1}{4}(|\mathscr{M}_{\rm uud\to uud}|^2+|\mathscr{M}_{\rm udu\to udu}|^2)+\frac{1}{4}|\mathscr{M}_{\rm udd\to udd}|^2\right]\times\\ & [f_{q1}f_{q2}f_{q3}(1-f_{q4})(1-f_{q5})(1-f_{q6})-f_{q4}f_{q5}f_{q6}(1-f_{q1})(1-f_{q2})(1-f_{q3})] \end{aligned} \tag{2}$$

A similar equation is given for the down-quark distribution. In the equations the degeneracy factors $g_{\rm G}=16$ for gluon and $g_{\rm Q}=6$ for up quark and the velocity $v_1=1$ for massless gluons and quarks. p_i stands for the four-momenta of the three initial partons with $i=1,2,3$ and of the three final partons with $i=4,5,6$. The distribution functions f_{gi} and f_{qi} depend on the position $\boldsymbol{r}$, the momentum $\boldsymbol{p}_i$ and the time t. The squared amplitudes for the 2-to-2 elastic scatterings, $|\mathscr{M}_{\rm gg\to gg}|^2$, $|\mathscr{M}_{\rm uu\to uu}|^2$ and $|\mathscr{M}_{\rm ud\to ud}|^2$ can be found in Refs. [16,17]. The spin- and color-averaged suqared amplitudes $|\mathscr{M}_{\rm ggg\to ggg}|^2$, $|\mathscr{M}_{\rm uuu\to uuu}|^2$, $|\mathscr{M}_{\rm uud\to uud}|^2$, $|\mathscr{M}_{\rm udu\to udu}|^2$ and $|\mathscr{M}_{\rm ddd\to ddd}|^2$ are used in the transport equations and the 3-to-3 elastic scatterings relate to a larger phase space than the 2-to-2 elastic scatterings. The transport equations are nonlinear equations of parton distribution functions.

When the squared four-momenta of gluon and/or quark propagators approach zero, the divergences of the squared amplitudes for the 2-to-2 and 3-to-3 elastic scatterings are encountered. They are removed while the propagators are regularized by a screening mass which is evaluated from the distribution function by a formula in Ref. [18]. The anisotropic parton momentum distributions are formed in the initial Au-Au collisions. Homogeneous approximation in space is made for the distributions. Such anisotropy can be eliminated by elastic scatterings among partons and thermal equilibrium is established as proved in Appendix B. Starting from the time $t_{\rm ini}=0.2{\rm fm}/c$, when the anisotropic momentum distributions are formed and ending at the time $t_{\rm iso}$ when the local momentum isotropy is established, the transport equations are solved. At $t_{\rm iso}$ the momentum distributions at the three angles $\theta=0°$, $45°$, $90°$ relative to one incoming gold beam direction overlap and can thus be fitted to the Jüttner distribution,

$$f(\boldsymbol{p},t_{\rm iso})=\frac{\lambda}{{\rm e}^{|\boldsymbol{p}|/T}-\lambda} \tag{3}$$

where for gluon matter[12], the temperature $T=0.75$GeV, fugacity $\lambda=0.065$ and $t_{\rm iso}=0.65{\rm fm}/c$ which leads to a thermalization time of $t_{\rm iso}-t_{\rm ini}=0.45{\rm fm}/c$; for quark matter[19], $T=0.59$GeV, $\lambda=0.04$, $t_{\rm iso}=2{\rm fm}/c$ and a corresponding thermalization time of $t_{\rm iso}-t_{\rm ini}=1.8{\rm fm}/c$.

4 Summary

The processes of ggg → ggg and qqq → qqq elastic scatterings are illustrated. The squared amplitude of the process with two four-gluon couplings is derived. The 3 → 3 parton elastic scatterings contribute to the evolution of the deconfined matter. The H-theorem proved shows that $2\rightarrow2$ and $3\rightarrow3$ elastic scatterings drive the deconfined matter described by the transport equations towards global thermal equilibrium. A thermalization time shorter (larger) than 1fm/c is obtained for the gluon (quark) matter initially created in the central Au-Au collisions at $\sqrt{s_{NN}}=200$GeV. If the quark-quark-antiquark and quark-antiquark-antiquark elastic scatterings are included[20], the thermalization time of quark matter given by the transport equation is shortened, but it is still larger than 1fm/c. Therefore, the elastic scatterings between gluons and quarks can be expected to further shorten the thermalization time.

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Appendix A

In Diagram $B_{\sim\sim}$ the colors, four-momenta, space-time indices, helicities and the polarization 4-vectors of the three initial gluons are labeled as a_i, p_i, μ_i, λ_i and $\epsilon_{\mu_i}(\lambda_i)$ with $i=1,2,3$ for gluons from left to right, respectively. The colors, four-momenta, space-time indices, helicities and the polarization 4-vectors of the three final gluons are labeled as b_j, p'_j, ν_j, λ'_j and $\epsilon_{\nu_j}(\lambda'_j)$ with $j=1,2,3$ for gluons from left to right, respectively. The virtual gluon has color g, space-time index λ and four-momentum p. In the appendix the summation implied by a repeated Greek letter runs from 0 to 3 and the summation implied by a repeated English letter runs from 1 to 8. The amplitude is

$$B_{\sim\sim\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\epsilon_{\mu_1}(\lambda_1)\epsilon_{\mu_2}(\lambda_2)\epsilon_{\mu_3}(\lambda_3)\epsilon^*_{\nu_1}(\lambda'_1)\epsilon^*_{\nu_2}(\lambda'_2)\epsilon^*_{\nu_3}(\lambda'_3)=\mathrm{i}\frac{g_\mathrm{s}^4}{p^2}\sum_{k=1}^{9}C_kT_k \ , \tag{A1}$$

with the quark-gluon coupling constant g_s and

$$C_1=f^{a_1a_2e}f^{a_3ge}f^{gb_3e'}f^{b_2b_1e'},$$
$$C_2=f^{a_1a_2e}f^{a_3ge}f^{gb_2e'}f^{b_3b_1e'},$$
$$C_3=f^{a_1a_2e}f^{a_3ge}f^{gb_1e'}f^{b_2b_3e'},$$
$$C_4=f^{a_1a_3e}f^{a_2ge}f^{gb_3e'}f^{b_2b_1e'},$$
$$C_5=f^{a_1a_3e}f^{a_2ge}f^{gb_2e'}f^{b_3b_1e'},$$
$$C_6=f^{a_1a_3e}f^{a_2ge}f^{gb_1e'}f^{b_2b_3e'},$$
$$C_7=f^{a_1ge}f^{a_3a_2e}f^{gb_3e'}f^{b_2b_1e'},$$
$$C_8=f^{a_1ge}f^{a_3a_2e}f^{gb_2e'}f^{b_3b_1e'},$$
$$C_9=f^{a_1ge}f^{a_3a_2e}f^{gb_1e'}f^{b_2b_3e'},$$

$$T_1 = (g_{\mu_1\mu_3}g_{\mu_2\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\nu_2}g_{\mu_2\mu_3}g_{\nu_3\nu_1}+g_{\mu_1\nu_1}g_{\mu_2\mu_3}g_{\nu_3\nu_2})W,$$

$$T_2 = (g_{\mu_1\mu_3}g_{\mu_2\nu_3}g_{\nu_2\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\nu_3}g_{\mu_2\mu_3}g_{\nu_2\nu_1}+g_{\mu_1\nu_1}g_{\mu_2\mu_3}g_{\nu_3\nu_2})W,$$

$$T_3 = (g_{\mu_1\mu_3}g_{\mu_2\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_3}g_{\nu_1\nu_2}-g_{\mu_1\nu_2}g_{\mu_2\mu_3}g_{\nu_3\nu_1}+g_{\mu_1\nu_3}g_{\mu_2\mu_3}g_{\nu_1\nu_2})W,$$

$$T_4 = (g_{\mu_1\mu_2}g_{\mu_3\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_2}g_{\mu_3\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\nu_2}g_{\mu_2\mu_3}g_{\nu_3\nu_1}+g_{\mu_1\nu_1}g_{\mu_2\mu_3}g_{\nu_3\nu_2})W,$$

$$T_5 = (g_{\mu_1\mu_2}g_{\mu_3\nu_3}g_{\nu_2\nu_1}-g_{\mu_1\mu_2}g_{\mu_3\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\nu_3}g_{\mu_2\mu_3}g_{\nu_2\nu_1}+g_{\mu_1\nu_1}g_{\mu_2\mu_3}g_{\nu_3\nu_2})W,$$

$$T_6 = (g_{\mu_1\mu_2}g_{\mu_3\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_2}g_{\mu_3\nu_3}g_{\nu_1\nu_2}-g_{\mu_1\nu_2}g_{\mu_2\mu_3}g_{\nu_3\nu_1}+g_{\mu_1\nu_3}g_{\mu_2\mu_3}g_{\nu_1\nu_2})W,$$

$$T_7 = (g_{\mu_1\mu_3}g_{\mu_2\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\mu_2}g_{\mu_3\nu_2}g_{\nu_3\nu_1}+g_{\mu_1\mu_2}g_{\mu_3\nu_1}g_{\nu_3\nu_2})W,$$

$$T_8 = (g_{\mu_1\mu_3}g_{\mu_2\nu_3}g_{\nu_2\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_1}g_{\nu_3\nu_2}-g_{\mu_1\mu_2}g_{\mu_3\nu_3}g_{\nu_2\nu_1}+g_{\mu_1\mu_2}g_{\mu_3\nu_1}g_{\nu_3\nu_2})W,$$

$$T_9 = (g_{\mu_1\mu_3}g_{\mu_2\nu_2}g_{\nu_3\nu_1}-g_{\mu_1\mu_3}g_{\mu_2\nu_3}g_{\nu_1\nu_2}-g_{\mu_1\mu_2}g_{\mu_3\nu_2}g_{\nu_3\nu_1}+g_{\mu_1\mu_2}g_{\mu_3\nu_3}g_{\nu_1\nu_2})W,$$

with $W=\epsilon_{\mu_1}(\lambda_1)\epsilon_{\mu_2}(\lambda_2)\epsilon_{\mu_3}(\lambda_3)\epsilon^*_{\nu_1}(\lambda'_1)\epsilon^*_{\nu_2}(\lambda'_2)\epsilon^*_{\nu_3}(\lambda'_3)$. The squared amplitude is

$$\sum_{\lambda_1\lambda_2\lambda_3\lambda'_1\lambda'_2\lambda'_3}|B_{\sim\sim\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\epsilon_{\mu_1}(\lambda_1)\epsilon_{\mu_2}(\lambda_2)\epsilon_{\mu_3}(\lambda_3)\epsilon^*_{\nu_1}(\lambda'_1)\epsilon^*_{\nu_2}(\lambda'_2)\epsilon^*_{\nu_3}(\lambda'_3)|^2=\frac{g_\mathrm{s}^8}{p^4}\sum_{i=1}^{9}\sum_{j=1}^{9}F_{ij}G_{ij} \ , \tag{A2}$$

where $F_{ij}=C_iC_j$ and $G_{ij}=T_iT^*_j$ form symmetric matrices

$$F=\begin{pmatrix}
648 & 324 & 324 & 324 & 162 & 162 & 324 & 162 & 162\\
324 & 648 & -324 & 162 & 324 & -162 & 162 & 324 & -162\\
324 & -324 & 648 & 162 & -162 & 324 & 162 & -162 & 324\\
324 & 162 & 162 & 648 & 324 & 324 & -324 & -162 & -162\\
162 & 324 & -162 & 324 & 648 & -324 & -162 & -324 & 162\\
162 & -162 & 324 & 324 & -324 & 648 & -162 & 162 & -324\\
324 & 162 & 162 & -324 & -162 & -162 & 648 & 324 & 324\\
162 & 324 & -162 & -162 & -324 & 162 & 324 & 648 & -324\\
162 & -162 & 324 & -162 & 162 & -324 & 324 & -324 & 648
\end{pmatrix}$$

$$G=\begin{pmatrix} 144 & 72 & 72 & 72 & 36 & 36 & 72 & 36 & 36 \\ 72 & 144 & -72 & 36 & 72 & -36 & 36 & 72 & -36 \\ 72 & -72 & 144 & 36 & -36 & 72 & 36 & -36 & 72 \\ 72 & 36 & 36 & 144 & 72 & 72 & -72 & -36 & -36 \\ 36 & 72 & -36 & 72 & 144 & -72 & -36 & -72 & 36 \\ 36 & -36 & 72 & 72 & -72 & 144 & -36 & 36 & -72 \\ 72 & 36 & 36 & -72 & -36 & -36 & 144 & 72 & 72 \\ 36 & 72 & -36 & -36 & -72 & 36 & 72 & 144 & -72 \\ 36 & -36 & 72 & -36 & 36 & -72 & 72 & -72 & 144 \end{pmatrix}$$

The elements F_{ij} are obtained with the help of the relationships $f^{acd}f^{bcd}=3\delta^{ab}$ and $f^{abe}f^{cde}f^{ace'}f^{bde'}=36$. Finally, the spin-and color-averaged squared amplitude is

$$\frac{1}{8}\frac{1}{2}\frac{1}{8}\frac{1}{2}\frac{1}{8}\frac{1}{2}\sum_{\lambda_1\lambda_2\lambda_3\lambda_1'\lambda_2'\lambda_3'}|B_{\sim\sim\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\epsilon_{\mu_1}(\lambda_1)\epsilon_{\mu_2}(\lambda_2)\epsilon_{\mu_3}(\lambda_3)\epsilon^*_{\nu_1}(\lambda_1')\epsilon^*_{\nu_2}(\lambda_2')\epsilon^*_{\nu_3}(\lambda_3')|^2=\frac{59049}{128}\frac{g_s^8}{(\sqrt{s})^4}\ , \tag{A3}$$

where $s=p^2$.

Appendix B

The H-theorem is the result of the conventional Boltzmann equation with 2-to-2 scatterings. We prove that the 3-to-3 scattering term in the transport equation (1) or (2) also allows the establishment of the H-theorem. With $f_{gi}=f_g(\boldsymbol{p}_i,\boldsymbol{r},t)$ where i takes the value from 1 to 6, we define

$$H(t)=\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{p}_1 f_g(\boldsymbol{p}_1,\boldsymbol{r},t)\ln f_g(\boldsymbol{p}_1,\boldsymbol{r},t)\ . \tag{B1}$$

The derivative of $H(t)$ with respect to time is derived by means of the transport equation for gluon matter

$$\begin{aligned}\frac{\partial H(t)}{\partial t} =& \int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{p}_1\frac{\partial f_{g1}}{\partial t}(\ln f_{g1}+1)=-\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{p}_1\boldsymbol{v}_1\cdot(\boldsymbol{\nabla}_{\boldsymbol{r}}f_{g1})(\ln f_{g1}+1)-\\ & g_{\mathrm{G}}\int \mathrm{d}\boldsymbol{r}\frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^32E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^32E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^32E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\\ & \frac{1}{2}|\mathscr{M}_{\mathrm{gg}\to\mathrm{gg}}|^2[f_{g1}f_{g2}(1+f_{g3})(1+f_{g4})-f_{g3}f_{g4}(1+f_{g1})(1+f_{g2})](\ln f_{g1}+1)-\\ & g_{\mathrm{G}}^2\int \mathrm{d}\boldsymbol{r}\frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^32E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^32E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^32E_4}\frac{\mathrm{d}^3p_5}{(2\pi)^32E_5}\frac{\mathrm{d}^3p_6}{(2\pi)^32E_6}\times\\ & (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\frac{1}{12}|\mathscr{M}_{\mathrm{ggg}\to\mathrm{ggg}}|^2(\ln f_{g1}+1)\times\\ & [f_{g1}f_{g2}f_{g3}(1+f_{g4})(1+f_{g5})(1+f_{g6})-f_{g4}f_{g5}f_{g6}(1+f_{g1})(1+f_{g2})(1+f_{g3})]\ .\end{aligned} \tag{B2}$$

The second term is separated into four terms in which one term is not changed and the other three terms have the exchanges $p_1\leftrightarrow p_2$, ($p_1\leftrightarrow p_3$, $p_2\leftrightarrow p_4$) and ($p_1\leftrightarrow p_4$, $p_2\leftrightarrow p_3$), respectively. The third term is separated into six terms in which one term is not changed, two terms have individually the exchanges $p_1\leftrightarrow p_2$ and $p_1\leftrightarrow p_3$, the other three terms have the exchanges ($p_1\leftrightarrow p_4$, $p_2\leftrightarrow p_5$, $p_3\leftrightarrow p_6$), ($p_1\leftrightarrow p_5$, $p_2\leftrightarrow p_4$, $p_3\leftrightarrow p_6$) and ($p_1\leftrightarrow p_6$, $p_2\leftrightarrow p_5$, $p_3\leftrightarrow p_4$), respectively. Since all the exchanges do not alter the integration results in the terms, we obtain

$$\begin{aligned}\frac{\partial H(t)}{\partial t} =& -\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{p}_1\boldsymbol{v}_1\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}(f_{g1}\ln f_{g1})-\\ & \frac{g_{\mathrm{G}}}{4}\int \mathrm{d}\boldsymbol{r}\frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^32E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^32E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^32E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\\ & \frac{1}{2}|\mathscr{M}_{\mathrm{gg}\to\mathrm{gg}}|^2[f_{g1}f_{g2}(1+f_{g3})(1+f_{g4})-f_{g3}f_{g4}(1+f_{g1})(1+f_{g2})]\ln\frac{f_{g1}f_{g2}}{f_{g3}f_{g4}}-\\ & \frac{g_{\mathrm{G}}^2}{6}\int \mathrm{d}\boldsymbol{r}\frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^32E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^32E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^32E_4}\frac{\mathrm{d}^3p_5}{(2\pi)^32E_5}\frac{\mathrm{d}^3p_6}{(2\pi)^32E_6}\times\\ & (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\frac{1}{12}|\mathscr{M}_{\mathrm{ggg}\to\mathrm{ggg}}|^2\times\\ & [f_{g1}f_{g2}f_{g3}(1+f_{g4})(1+f_{g5})(1+f_{g6})-f_{g4}f_{g5}f_{g6}(1+f_{g1})(1+f_{g2})(1+f_{g3})]\ln\frac{f_{g1}f_{g2}f_{g3}}{f_{g4}f_{g5}f_{g6}}\ .\end{aligned} \tag{B3}$$

Since the distribution function is zero at $r\to\infty$, the first term equals zero. In practical calculations the approximation $1+f_i\approx 1$ is taken. Then due to $(x-y)\ln\frac{x}{y}>0$,

$$\frac{\partial H(t)}{\partial t}\leqslant 0. \tag{B4}$$

$H(t)$ always decreases with time increasing until it is independent of time. When $H(t)$ does not rely on time, $\frac{\partial H(t)}{\partial t}=0$ and the distribution function is independent of time. To ensure $\frac{\partial H(t)}{\partial t}=0$,

$$f_{g1}f_{g2}-f_{g3}f_{g4}=0, \tag{B5}$$

$$\ln f_{g1}+\ln f_{g2}=\ln f_{g3}+\ln f_{g4}, \tag{B6}$$

for the $2\to 2$ elastic scatterings and

$$f_{g1}f_{g2}f_{g3}-f_{g4}f_{g5}f_{g6}=0, \tag{B7}$$

$$\ln f_{g1}+\ln f_{g2}+\ln f_{g3}=\ln f_{g4}+\ln f_{g5}+\ln f_{g6}, \tag{B8}$$

for the $3\to 3$ elastic scatterings. The conserved quantity $\ln f_g$ in elastic scatterings must relate to other conserved quantities like energy and momentum. Then

$$\ln f_g=a+\boldsymbol{b}\cdot\boldsymbol{p}+cE \tag{B9}$$

where a and c are constants and $\boldsymbol{b}$ is a constant vector. We obtain the distribution function

$$f_g=\mathrm{e}^{a+\boldsymbol{b}\cdot\boldsymbol{p}+cE}, \tag{B10}$$

which is a thermal distribution. We conclude that the gluon matter described by the transport Eq. (1) must evolve into a thermal state. For quark matter with equal up and down quark distribution functions, the above proof applies to the transport Eq. (2) to draw the same conclusion.

3到3的部分子弹性散射和早期热平衡化*

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摘要　研究了三胶子弹性散射 ggg → ggg 和三夸克弹性散射 qqq → qqq. 证明了包含 3 → 3 弹性散射的输运方程的 H 定理. 输运方程给出了胶子物质热平衡化时间短和夸克物质热平衡化时间长的结果.

关键词　三胶子散射　三夸克散射　输运方程　热平衡化

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