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# Exact Bound Solution of the Klein-Gordon Equation and the Dirac Equation with Rosen-Morse II Potential*

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**Abstract** In this paper, the relativistic Rosen-Morse II potential is investigated by solving the Klein-Gordon and the Dirac equations with equal attractive scalar $s(\boldsymbol{r})$ and repulsive vector $v(\boldsymbol{r})$ potentials. The exact energy equations of the bound state are obtained by the method of supersymmetric and shape invariance. Finally, a kind of special potential about Rosen-Morse II potential is discussed.

**Key words** Klein-Gordon equation, Dirac equation, bound state, exact solution, Rosen-Morse II potential

## 1 Introduction

It is well known that exact energy eigenvalues of the bound state play an important role in quantum mechanics. Usually, we describe quantum mechanical behavior of a particle within non-relativistic frame. However, when the particle is under strong field, especially for a strong coupling system, relativistic effect could become important. Many researchers have solved the Klein-Gordon and the Dirac equations with equal scalar and vector potentials, and given the bound state solutions of particles in some typical potential fields[1—5].

With the exception of the Harmonic Oscillator and Coulomb potential, Rosen-Morse potential is an exact solvable potential; this potential is of possible interest to quark physics so far as it captures the essentials of the QCD quark-gluon dynamics[6]. Rosen-Morse II potential[7] is $V(r)$, i.e.,

$$V(r)=\frac{B^2}{A(A+\alpha)}+A(A+\alpha)\tanh^2(\alpha r)+2B\tanh(\alpha r), \tag{1}$$

where $A$, $B$, and $\alpha$ are constant, and $\alpha$ is a regular scale factor. The authors[8] have solved the Schrödinger equation for more general Rosen-Morse II potential.

Unfortunately, so far the solutions of the Klein-Gordon and the Dirac equations with Rosen Morse II potential have not yet been solved. In this paper, we assume that the scalar potential $s(\boldsymbol{r})$ equals to the vector potential $v(\boldsymbol{r})$ and solve the Klein-Gordon as well as the Dirac equations with Rosen-Morse II potential by using supersymmetric and shape invariance method, and obtain the energy equations of the bound state. In the case of $\tanh\mu = B/A(A+\alpha)$ and $d=1/\alpha$, a kind of special potential about Rosen-Morse II potential is discussed, and we obtain such a result that the harmonic potential is the limit of the potential, and when $B/A(A+\alpha)=0$, the potential becomes $V_0\tanh^2(r/d)$ potential, with which Qiang[4] has solved the Klein-Gordon equation and the Dirac equation. Our result is similar to Qiang's.

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## 2 Bound-state solution of the Klein-Gordon equation

For simplicity, the atomic units ($\hbar = c = 1$) are taken, the Klein-Gordon equation with the special potential is[9]

$$[p^2-(E-v(\boldsymbol{r}))^2]\,\psi=-[m+s(\boldsymbol{r})]^2\,\psi\ , \tag{2}$$

where $p$ is the momentum operator, $E$ and $m$ are the energy and mass of the particle, $v(\boldsymbol{r})$ is the vector potential, and $s(\boldsymbol{r})$ is the scalar potential, they are equal to $V(r)/2$. To find the corresponding relativistic quantum mechanical behavior, we solve Eq. (2) in spherical coordinate. The radial part is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r))+\Big[(E^2-m^2)-(E+m)V(r)-\frac{l(l+1)}{r^2}\Big]R(r)=0\,. \tag{3}$$

When $r$ tends towards infinity, the radial wave function is equal to zero, and $R(r)$ is finite at $r=0$, so we can set

$$R(r)r=u(r)\ . \tag{4}$$

For $s$ wave, $l=0$, using the relation $\lambda=E^2-m^2$, accordingly, Eq. (3) has been transformed into

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}-(E+m)\bigg(\frac{B^2}{A(A+\alpha)}+A(A+\alpha)\tanh^2(\alpha r)+2B\tanh(\alpha r)\bigg)u(r)=\lambda u(r). \tag{5}$$

To solve Eq. (5) we apply the supersymmetric quantum mechanics and shape invariance approach[10, 11]. The ground state wave function $u_0(r)$ can be written in the fashion of

$$u_0(r)=\exp\left(-\int W(r)\mathrm{d}r\right), \tag{6}$$

where $W(r)$ is super potential in the super symmetric quantum mechanics. Substituting the ground state function into Eq. (5), we obtain an equation about the ground state energy, namely

$$W^2-\frac{\mathrm{d}W}{\mathrm{d}r}\ =\ (E+m)\Bigg(A(A+\alpha)\times\left(\tanh(\alpha r)+\frac{B}{A(A+\alpha)}\right)^2\Bigg)-\lambda_0\,, \tag{7}$$

where $\lambda_0$ is the ground state energy. Eq. (7) is a Riccati equation. The corresponding super potential is set

$$W(r)=Q_1\tanh(\alpha r)+Q_2\,. \tag{8}$$

Substituting the expression Eq. (8) into Eq. (7), we obtain three equations as follows:

$$-\frac{B^2(E+m)}{A(A+\alpha)}-\alpha Q_1+Q_2^2+\lambda_0=0\,, \tag{9}$$

$$-2B(E+m)+2Q_1Q_2=0\,, \tag{10}$$

$$-A(A+\alpha)(E+m)+\alpha Q_1+Q_1^2=0\,. \tag{11}$$

Because of the limit of radial wave function boundary condition, we solve Eq. (10) and Eq. (11), and obtain

$$Q_1=\alpha\left\{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{A}{\alpha^2}(A+\alpha)(E+m)}\right\}, \tag{12}$$

$$Q_2=\frac{B(E+m)}{Q_1}\,. \tag{13}$$

Then the ground state function can be expressed as

$$u_0(r)\ =\ \exp\left\{\frac{2B(E+m)}{-\alpha+\sqrt{\alpha^2+4A(A+\alpha)(E+m)}}\right\}\times\cosh(\alpha r)^{\frac{1}{2}-\frac{\sqrt{\alpha^2+4A(A+\alpha)(E+m)}}{2\alpha}}. \tag{14}$$

By solving Eq. (9), the corresponding ground state energy is obtained by

$$\lambda_0=\frac{B^2(E+m)}{A(A+\alpha)}+\alpha Q_1-\left(\frac{B(E+m)}{Q_1}\right)^2. \tag{15}$$

Uniting Eq. (8), Eqs. (12) and (13), we can obtain the partner potentials

$$V_+(r)\ =\ Q_1^2+\left(\frac{B(E+m)}{Q_1}\right)^2+B(E+m)\tanh(\alpha r)+(\alpha-Q_1)Q_1\mathrm{sech}^2(\alpha r), \tag{16}$$

$$V_-(r)\ =\ Q_1^2+\left(\frac{B(E+m)}{Q_1}\right)^2+B(E+m)\tanh(\alpha r)-(\alpha+Q_1)Q_1\mathrm{sech}^2(\alpha r). \tag{17}$$

When $a_0=Q_1$, $a_1=f(a_0)=Q_1-\alpha$, $V_+(r)$ and $V_-(r)$ satisfy the following relationship,

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1), \tag{18}$$

where $R(a_1)\ =\ Q_1^2+\left(\frac{B(E+m)}{Q_1}\right)^2-(Q_1-\alpha)^2-\left(\frac{B(E+m)}{Q_1-\alpha}\right)^2$. Eq. (18) shows that the partner potential $V_-(r)$ is a shape-invariant potential. The energy spectra of the potential, hence the shape-invariant potential are given by

$$\begin{aligned}\lambda_0^- &= 0\\ \lambda_n^- &= \sum_{i=1}^{n} R(a_i) = Q_1^2 + \left(\frac{B(E+m)}{Q_1}\right)^2 - \\ &\quad (Q_1-n\alpha)^2 - \left(\frac{B(E+m)}{Q_1-n\alpha}\right)^2 . \end{aligned} \tag{19}$$

Thus we can have the value of $\lambda_n$ in Eq. (5)

$$\begin{aligned}\lambda_n &= \frac{B^2(E+m)}{A(A+\alpha)} - \left(\frac{B(E+m)}{Q_1-n\alpha}\right)^2 + (2n+1)\alpha Q_1 - \\ &\quad (n\alpha)^2 = -(n\alpha)^2 + \frac{B^2(E+m)}{A(A+\alpha)} + 2\left(n+\frac{1}{2}\right)\times \\ &\quad \alpha^2\left\{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{A}{\alpha^2}(E+m)(A+\alpha)}\right\} - \\ &\quad \frac{\frac{B^2(E+m)^2}{\alpha^2}}{\left(\left(\frac{1}{2}+n\right)-\sqrt{\frac{1}{4}+\frac{A}{\alpha^2}(E+m)(A+\alpha)}\right)^2} . \end{aligned} \tag{20}$$

## 3　Bound states solution of Dirac equation

The Dirac equation with scalar potential $s(\boldsymbol{r})$ and vector potential $v(\boldsymbol{r})$ is ($\hbar = c = 1$)

$$\{\alpha\cdot p + \beta[m+s(\boldsymbol{r})]\}\Psi = [E - v(\boldsymbol{r})]\Psi . \tag{21}$$

In relativistic quantum mechanics, the complete conservative quantity set of a particle in a central field can be taken to be ($H$, $K$, $J^2$, $J_z$), the eigenfunctions of which are[12]

$$\begin{cases}\Psi = \dfrac{1}{r}\begin{pmatrix} F(r)\phi^A_{jm_j} \\ \mathrm{i}G(r)\phi^B_{jm_j}\end{pmatrix} \qquad K = \left(j+\dfrac{1}{2}\right), \\[2ex] \Psi = \dfrac{1}{r}\begin{pmatrix} F(r)\phi^B_{jm_j} \\ \mathrm{i}G(r)\phi^A_{jm_j}\end{pmatrix} \qquad K = -\left(j+\dfrac{1}{2}\right),\end{cases} \tag{22}$$

where

$$\begin{cases}\phi^A_{jm_j} = \dfrac{1}{\sqrt{2l+1}}\begin{pmatrix}\sqrt{l+m+1}\,Y_{lm} \\ \sqrt{l-m}\,Y_{l(m+1)}\end{pmatrix}, \\[2ex] \phi^B_{jm_j} = \dfrac{1}{\sqrt{2l+3}}\begin{pmatrix}-\sqrt{l+1-m}\,Y_{(l+1)m} \\ \sqrt{l+1+m+1}\,Y_{(l+1)(m+1)}\end{pmatrix},\end{cases}$$

and $K$ is the spin-orbit coupling angular momentum defined as $K = \pm\left(j+\frac{1}{2}\right)$ for $l = j\pm\frac{1}{2}$. The terms $\Psi$ in Eq. (21) are replaced by Eq. (22), accordingly, we can obtain the radial part of the Dirac equation,

$$\begin{cases}\dfrac{\mathrm{d}F}{\mathrm{d}r} - \dfrac{K}{r}F = (m+E)G, \\[2ex] \dfrac{\mathrm{d}G}{\mathrm{d}r} + \dfrac{K}{r}G = (m-E+V(r))F.\end{cases} \tag{23}$$

By eliminating $G(r)$, a second-order differential equation for $F(r)$ is obtained.

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2} - \frac{K(K-1)}{r^2}F + \big((E^2-m^2)-(E+m)V(r)\big)F = 0 . \tag{24}$$

For s wave, i.e., $K = 1$. Eq. (24) becomes

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2} + \big((E^2-m^2)-(E+m)V(r)\big)F = 0 , \tag{25}$$

Eq. (25) is identical with Eq. (5). Hence, we can obtain the energy equation with Rosen-MorseⅡ potential, i.e.

$$\begin{aligned}\lambda_n &= -(n\alpha)^2 + \frac{B^2(E+m)}{A(A+\alpha)} + 2\left(n+\frac{1}{2}\right)\times \\ &\quad \alpha^2\left\{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{A}{\alpha^2}(E+m)(A+\alpha)}\right\} - \\ &\quad \frac{\frac{B^2(E+m)^2}{\alpha^2}}{\left(\left(\frac{1}{2}+n\right)-\sqrt{\frac{1}{4}+\frac{A}{\alpha^2}(E+m)(A+\alpha)}\right)^2} . \end{aligned} \tag{26}$$

## 4　Discussion

If we take $\tanh\mu = B/A(A+\alpha)$ and $d = 1/\alpha$, then Eq. (1), namely the potential is

$$V(r) = V_0\cosh^2\mu\left\{\tanh\left(\frac{r}{d}\right)+\tanh(\mu)\right\}^2 , \tag{27}$$

where $V_0 = A(A+\alpha) - \dfrac{B^2}{A(A+\alpha)}$, and $V_0$ is positive when $A(A+\alpha) > B$. This potential field has its minimum value at $r = 0$. As $r$ is increased positive, the potential rises to an asymptotic value $V_0\mathrm{e}^{2\mu}$ for $r\to+\infty$. When a particle of the order of a few hundred MeV moves in the potential valley, some of the allowed energies will be discrete values. After a series of calculation we obtain the bound–state energy spectrum, namely,

$$\lambda_n = -(n\alpha)^2 + (E+m)V_0 \sinh^2\mu + 2\left(n+\frac{1}{2}\right)\times \alpha^2\left\{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{V_0\cosh^2\mu}{\alpha^2}(E+m)}\right\} - \frac{\frac{V_0^2(E+m)^2\sinh^2[2\mu]}{4\alpha^2}}{\left(\left(\frac{1}{2}+n\right)-\sqrt{\frac{1}{4}+\frac{V_0\cosh^2\mu}{\alpha^2}(E+m)}\right)^2}. \quad (28)$$

When the particle moves in the bottom of the potential valley, its radial coordinate $r$ is far smaller than the scale factor $d$, we expand Eq. (1) for $r$ small near the potential minimum at $r=0$,

$$V(r)\approx (r/d)^2 V_0 \mathrm{sech}^2\mu + \cdots; \qquad r \ll d. \quad (29)$$

Thus we only take the first term into account, the form of potential energy is similar to that of harmonic oscillator. Consequently, the frequency of harmonic oscillator is $\omega/2\pi$, where $\omega = \sqrt{2V_0/\mu d^2\cosh^2\mu}$, as well as elastic coefficient $k$ is $k = 2V_0\mathrm{sech}^2\mu/d^2$. Ref. [2] gives the energy equations of a relativistic harmonic oscillator potential

$$\begin{cases} (4n+3+2l)\sqrt{k}-(E-m)\sqrt{2(m+E)}=0, \\ \text{(Klein-Gordon)} \\ (4n+3+2K)\sqrt{k}-(E-m)\sqrt{2(m+E)}=0, \\ \text{(Dirac)} \\ \qquad (n=0,1,2,3,\cdots). \end{cases} \quad (30)$$

For $s$ wave, i.e., $l=0$ and $K=0$, Eq. (30) is uniform equation

$$(4n+3)\sqrt{k}-(E-m)\sqrt{2(E+m)}=0, (n=0,1,2,\cdots). \quad (31)$$

Then

$$E=\frac{1}{12}\left(4m+\frac{2^{14/3}m^2}{q}+2^{4/3}q\right),$$

where

$$q = (-32m^3+27k(3+4n)^2+ 3\sqrt{3}\sqrt{k(3+4n)^2(-64m^3+27k(3+4n)^2)^{1/3}}.$$

Thus it is necessary to discuss energy equation when $r \ll d$. Now, we express Eq. (28) into

$$1+2n+d\sqrt{E+m}\times g=0, \quad (32)$$

where

$$g = \sqrt{V_0 e^{2\mu}+m-E}+\sqrt{V_0 e^{-2\mu}+m-E} - \frac{2}{d\sqrt{m+E}}\sqrt{\frac{1}{4}+d^2(m+E)V_0\cosh^2\mu}.$$

For the particle moving in the bottom, its $(E-m)$ is far smaller than its potential maximum $V_0 e^{2\mu}$. On the other hand, when $[d^2(m+E)V_0\cosh^2\mu]$ is considerably larger than $\frac{1}{4}$, the last term in $g$ becomes $2\sqrt{V_0}\cosh\mu$. And then

$$g\approx\frac{(m+E)\cosh\mu}{\sqrt{V_0}}. \quad (33)$$

Substituting Eq. (33) into Eq. (32), we obtain the same energy equation to harmonic oscillator formula, i.e., Eq. (31).

When $d=1/\alpha$ and $B/A(A+\alpha)=0$, then the potential becomes the $V_0\tanh^2(r/d)$ potential[4],

$$V(r)=V_0\tanh^2(r/d), \quad (34)$$

which is the symmetric case of Eq. (27). Solution for the potential is

$$2d\sqrt{(m+E)(V_0+m-E)} - \sqrt{1+4d^2(m+E)V_0}=-1-2n. \quad (35)$$

Qiang has obtained the bound-state solution of Klein-Gordon equation and Dirac equation for $V_0\tanh^2(r/d)$ potential, our conclusion is similar.

## 5 Conclusion

In this paper, by applying supersymmetric and shape invariance technique, we have obtained exact solutions of bound states for the Klein-Gordon and the Dirac equations both with equal scalar Rosen Morse II potential and vector Rosen Morse II potential. Energy Eq. (20) is the relativistic description of a particle of spin 0 such as K-meson and μ meson. Eq. (26) is relativistic description for a particle of spin 1/2 such as electron, neutron, proton and hyperon. Eq. (27) is a special case of Rosen Morse II potential.

On the other hand, theoretical prediction of many properties of atoms or molecules requires the knowledge of continuous states and the phase shifts, so continuous states for the Rosen Morse Ⅱ potential is very important. We will further try to apply RMF method[13] to calculate the scattering amplitude and the phase shift for Rosen-Morse Ⅱ potential, this would bring out some complications in solving the equations, as well as new and interesting physics.

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# Rosen-Morse Ⅱ 势函数的 Klein-Gordon 方程和 Dirac 方程束缚态的精确解*


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**摘要**　在标量势和矢量势相等的情形, 研究了 Rosen-Morse Ⅱ 势的相对论效应, 应用超对称和形状不变势方法通过求解 Klein-Gordon 方程和 Dirac 方程得到了束缚态能量本征值, 最后, 讨论了 Rosen-Morse Ⅱ 势的一种特殊情况.



**关键词**　Klein-Gordon 方程　Dirac 方程　束缚态　精确解　Rosen-Morse Ⅱ 势函数



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