Boundary Reflection Factor of the Supersymmetric Sinh-Gordon Model

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Abstract The supersymmetric sinh-Gordon model on a half-line with integrable boundary conditions is considered perturbatively to verify conjectured exact reflection factors to one loop order. Propagators for the boson and fermion fields restricted to a half-line contain several novel features and are developed as prerequisites for the calculations.

Key words supersymmetry, sinh-Gordon model, reflection factor

1 Introduction

Given an interesting field theory, it is traditional to develop and examine its supersymmetric extensions. In four dimensions, supersymmetric field theories provide the prime examples of situations in which quantities of physical interest may be calculated exactly. For this reason they are an important source of ideas and intuition. However, in two dimensions, for nonlinear models, the two requirements of supersymmetry and integrability do not always sit easily together. There are many examples of two-dimensional models which are both integrable and supersymmetric; for a selection see Ref. [1]. The supersymmetric version of the sine-Gordon model was introduced in Ref. [2]; Shankar and Witten[3] constructed its exact *S*-matrix which was subsequently further explored by Schoutens[4].

If a field theory is restricted to a half-line by integrable boundary conditions then it turns out that supersymmetry is further constrained, and more restrictive. In the case of the sine-Gordon model, it was pointed out by Inami, Odake and Zhang[5] that only two isolated boundary conditions are compatible with both supersymmetry and integrability. This is a striking and surprising result since without supersymmetry Ghoshal and Zamolodchikov[6] had earlier pointed out that there should be a two parameter family of nonlinear boundary conditions compatible with integrability. More recently, using general arguments, exact reflection matrices for the breathers and their fermionic partners within the N=1 supersymmetric sine-Gordon theory have been conjectured[7].

In this paper, we will examine supersymmetric sinh-Gordon 'theory restricted to a half-line by integrable boundary conditions. The paper is organised as follows: in section two, we summarise the main features of the model and describe the boson and fermion propagators; the construction of the supersymmetric reflection factors for the two allowed boundary conditions are presented in section three together with reasons for deviating from the suggestions made by Moriconi and Schoutens; in the final section we check the fermion reflection factors agree with the perturbation expansion up to second order in the bulk coupling constant.

2 The supersymmetric sinh-Gordon model with one boundary

To establish the conventions we shall use, it is convenient to start with the supersymmetric sinh-Gordon model on a half line described by the Lagrangian density

$$S=\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{0}\mathrm{d}x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi-\overline{\psi}\ \mathrm{i}\gamma^\mu\partial_\mu\psi+m\overline{\psi}\psi\cosh\frac{\beta\phi}{\sqrt{2}}\right]-$$

$$\int_{-\infty}^{\infty}\mathrm{d}t\left[\pm\frac{2m}{\beta^2}\cosh\frac{\beta\phi}{\sqrt{2}}\mp\frac{1}{2}\overline{\psi}\psi\right] \quad . \tag{2.1}$$

where β is a real coupling constant, m is a mass parameter, ϕ is a real scalar field and ψ is a two-component Majorana fermion. The action is supersymmetric if and only if the two components of the parameter ε satisfy $\varepsilon_1=\mp\varepsilon_2$, which confirms that only half the supersymmetry of the bulk theory is preserved in the presence of boundary conditions. The boundary conditions for the fields at $x=0$ follow from Eq. (2.1)

$$\partial_x\phi=\mp\frac{\sqrt{2}m}{\beta}\sinh\frac{\beta\phi}{\sqrt{2}},\qquad \psi_1=\pm\psi_2. \tag{2.2}$$

We shall refer to these two boundary conditions as $\mathrm{BC}^{\pm}$, the $\pm$ corresponding to the signs relating the fermion components in each of the two cases given in Eq. (2.2).

The construction of the boson propagator for the sinh-Gordon model in the presence of integrable boundary conditions was given in Ref. [8], but the fermion propagator is constructed here for the first time. In the supersymmetric case we have just two kinds of boundary condition preserving both supersymmetry and integrability. The boson propagators corresponding to these are given by

$$G^{\pm}(x,t;x',t')=\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{\omega^2-k^2-m^2+\mathrm{i}\varepsilon}\left[\mathrm{e}^{\mathrm{i}k(x-x')}+K_{\mathrm{b}}^{\pm}(k)\mathrm{e}^{-\mathrm{i}k(x+x')}\right]. \tag{2.3}$$

The coefficients of the reflected term in the integrand of Eq. (2.3) correspond to the 'classical' reflection factors of the model linearised about the ground state solution $\phi=0$,

$$K_{\mathrm{b}}^{\pm}(k)=\frac{\mathrm{i}k\pm m}{\mathrm{i}k\mp m}=\frac{\mathrm{i}\ \sinh\theta\pm1}{\mathrm{i}\ \sinh\theta\mp1} \quad . \tag{2.4}$$

In Eq.(2.4), the second form of the expression refers to the on-shell reflection factor for a particle with rapidity θ for which $k=m\cosh\theta$.

We are familiar with the usual expression for a fermion propagator on the whole line. In the presence of the boundary we need to modify the standard fermion propagator, ensuring not only that it performs as a propagator in the bulk but also that it respects the fermion part of the boundary conditions Eq. (2.2). In two dimensions, with our choice of γ-matrices, the expression for the fermion propagators is the following:

$$S_{\mathrm{F}}^{\pm}(x,t;x',t')=\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{\omega^2-k^2-m^2+\mathrm{i}\varepsilon}\cdot$$

$$\left[\begin{pmatrix} m & -\mathrm{i}(\omega+k) \\ \mathrm{i}(\omega-k) & m \end{pmatrix}\mathrm{e}^{\mathrm{i}k(x-x')}\pm\frac{\omega}{\mathrm{i}k\mp m}\begin{pmatrix} \omega-k & -\mathrm{i}m \\ \mathrm{i}m & \omega+k \end{pmatrix}\mathrm{e}^{\mathrm{i}k(x+x')}\right]. \quad (2.5)$$

From the expression Eq. (2.5), it is natural to take the ‘classical’ fermion reflection factors to be given by

$$K_{\mathrm{f}}^{\pm}=\pm\frac{\omega}{\mathrm{i}k\mp m}=\frac{\cosh\theta}{\mathrm{i}\ \sinh\theta\mp 1}\ , \quad (2.6)$$

and as before, the second expression refers to the the on-shell reflection factors.

3 The construction of the reflection factorsfor the supersymmetric theory

Moriconi and Schoutens[7] assumed that the reflection matrix can be factorised in the following form,

$$R(\theta)=R_{\mathrm{b}}(\theta)R_{\mathrm{s}}(\theta)\ . \quad (3.1)$$

Here, θ is the rapidity of the reflecting particle, $R_{\mathrm{b}}(\theta)$ would be the reflection matrix for the bosonic part of the theory in the absence of fermions, and R_{s} is the supersymmetric part and have the form

$$K_{\mathrm{b}}^{\pm}(\theta)=R_{\mathrm{b}}^{\pm}(\theta)Z^{\pm}(\theta)\ \cosh\left(\frac{\theta}{2}\pm\frac{\mathrm{i}\pi}{4}\right)\ ,\ K_{\mathrm{f}}^{\pm}(\theta)=R_{\mathrm{b}}^{\pm}(\theta)Z^{\pm}(\theta)\ \cosh\left(\frac{\theta}{2}\mp\frac{\mathrm{i}\pi}{4}\right)\ . \quad (3.2)$$

In the classical limit the complete reflection matrix must match the boson and fermion classical factors. This requires particular classical limits for $Z^{\pm}(\theta)$, namely

$$Z^{\pm}(\theta)\to\frac{1}{\cosh\left(\frac{\theta}{2}\pm\frac{\mathrm{i}\pi}{4}\right)}. \quad (3.3)$$

In addition, the factor $Z^{\pm}(\theta)$ is constrained by the requirements of unitarity and by boundary crossing unitarity. Given the classical limits Eq. (3.3) it is natural to set $Z^{\pm}=\widetilde{Z}^{\pm}/\cosh\left(\frac{\theta}{2}\pm\frac{\mathrm{i}\pi}{4}\right)$, then the solutions to the two conditions are

$$\widetilde{Z}^{-}(\theta)=\exp\left[\frac{\mathrm{i}}{2}\int_0^{\infty}\frac{\mathrm{d}t}{t}\frac{\sinh\rho t\,\sinh(1+\rho)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right], \tag{3.4}$$

and

$$\widetilde{Z}^{+}(\theta)=\exp\left[-2\mathrm{i}\int_0^{\infty}\frac{\mathrm{d}t}{t}\frac{\sinh\frac{\rho t}{2}\,\sinh\frac{1}{2}(1+\rho)t}{\cosh^2\frac{t}{2}}\sin\frac{\theta t}{\pi}\right]\exp\left[\frac{\mathrm{i}}{2}\int_0^{\infty}\frac{\mathrm{d}t}{t}\frac{\sinh\rho t\,\sinh(1+\rho)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \tag{3.5}$$

Notice that these are not quite the same as the proposals made in Ref. [7] since Moriconi and Schoutens took the view that the classical limit of a free boson reflection factor should be unity; an assumption which is not generally valid, as we have seen.

Ghoshal[9] has calculated a formula for the quantum reflection matrix for the breather states of the sine-Gordon model. The reflection factor for the sinh-Gordon model is presumed to be deduced from the lightest breather reflection factor in the sine-Gordon theory by analytic continuation in the coupling constant (replacing β by iβ), leading to the expression

$$R_{\mathrm{b}}(\theta)=\frac{(2-B/2)(1)(1+B/2)}{(1-E)(1+E)(1-F)(1+F)}, \tag{3.6}$$

where $B=2\beta^2/(\beta^2+4\pi)$ and the coupling dependent functions E and F also depend on the boundary parameters introduced via the boundary potential (E and F are related to the parameters η and ϑ in Ghoshal's notation by $E=\eta B/\pi$, $F=\mathrm{i}\vartheta B/\pi$). In the supersymmetric theory, we consider the boundary conditions Eq.(2.2) for which F=0. On the other hand, an expression for E has been found by comparing two independent calculations of the boundary breather spectrum[10]. This translates in the present situation with two possible boundary conditions Eq. (2.2) to

$$\mathrm{BC}^{+}: E=0, \qquad \mathrm{BC}^{-}: E=2(1-B/2). \tag{3.7}$$

Thus we take

$$R_{\mathrm{b}}^{+}=\frac{(1+B/2)(2-B/2)}{(1)^3}, \tag{3.8}$$

and

$$R_{\mathrm{b}}^{-}=\frac{(1+B/2)(2-B/2)(1+B)(1-B)}{(1)}. \tag{3.9}$$

Notice that Eq. (3.9) contains the bound-state pole (in the factor $(1-B)$ at $\theta=\mathrm{i}(1-B)\pi/2$, whereas Eq. (3.8) contains no bound states. The suggestions made by Moriconi and Schoutens were different but for

comparison we list them here:

$$R_{\mathrm{b}}^{+}=\frac{(2-B/2)}{(1+B/2)(1)}, \tag{3.10}$$

and

$$R_{\rm b}^{-}=(1)(1+B/2)(2-B/2) \tag{3.11}$$

corresponding to E=B/2 and E=2, respectively. To order order β^2 we have identical expansions for Eqs. (3.8) and (3.10),

$$R_{\rm b}^{+}\sim\left(\frac{{\rm i}\ \sinh\theta+1}{{\rm i}\ \sinh\theta-1}\right)\left[1-\frac{{\rm i}\beta^2}{8}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right] \tag{3.12}$$

as indeed we do for Eq.(3.9) and (3.11),

$$R_{\rm b}^{-}\sim\left(\frac{{\rm i}\ \sinh\theta-1}{{\rm i}\ \sinh\theta+1}\right)\left[1-\frac{{\rm i}\beta^2}{8}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right]\ . \tag{3.13}$$

To conclude this section we shall prepare the way for comparing the reflection factors with low order perturbation theory by giving their expansions to order β^2. This is straightforward apart from a couple of complicated integrals arising from the Z-factors. For example, setting $\rho\sim\rho_0\beta^2/8\pi$, we have

$$\tilde{Z}^{+}(\theta)=1-\rho_0\frac{{\rm i}\beta^2}{16\pi}\left[\frac{2\theta}{\cosh\theta}-\pi\ \sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right]\ . \tag{3.14}$$

Combining, Eqs. (3.12) and (3.14) we deduce expressions for the supersymmetric reflection factors corresponding to the boundary conditions ${\rm BC}^{+}$ to order β^2:

$$K_{\rm b}^{+}(\theta)\sim\frac{{\rm i}\ \sinh\theta+1}{{\rm i}\ \sinh\theta-1}M^{+}(\theta),\quad K_{\rm f}^{+}(\theta)\sim\frac{\cosh\theta}{{\rm i}\ \sinh\theta-1}M^{+}(\theta),$$

$$M^{+}(\theta)=1-\frac{{\rm i}\beta^2}{16\pi}\left((2-\rho_0)\pi\ \sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)+\frac{2\rho_0\theta}{\cosh\theta}\right)\ . \tag{3.15}$$

In a similar manner, the expansions of the reflection factors corresponding to the boundary conditions ${\rm BC}^{-}$ are:

$$K_{\rm b}^{-}(\theta)\sim\frac{{\rm i}\ \sinh\theta-1}{{\rm i}\ \sinh\theta+1}M^{-}(\theta),\quad K_{\rm f}^{-}(\theta)\sim\frac{{\rm i}\ \sinh\theta-1}{\cosh\theta}M^{-}(\theta),$$

$$M^{-}(\theta)=1-\frac{{\rm i}\beta^2}{16\pi}\left((2-\rho_0)\pi\ \sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)-\frac{2\rho_0\theta}{\cosh\theta}\right)\ . \tag{3.16}$$

4　The fermion reflection factors

Using a path integral formalism and perturbation theory, one can obtain an expression for the generating functional for the supersymmetric sinh-Gordon model up to one-loop order. Subsequently the boson and fermion two-point functions[11] can be evaluated up to the same order in order to obtain the fermion reflection factor. We first calculate the fermion reflection factor corresponding to the case ${\rm BC}^{+}$. The contribution corresponds

$$-\frac{\rm i}{4}m\beta^2\int_{-\infty}^{+\infty}{\rm d}t''\int_{-\infty}^{0}{\rm d}x''S_{\rm F}^{+}(x,t;x'',t'')G^{+}(x'',t'';x'',t'')S_{\rm F}^{+}(x'',t'',x',t')\ , \tag{4.1}$$

Inserting the fermion and boson propagators in expression (4.1), and performing the integrals, we remain with the intergal over ω from which we extract the fermion reflection factor. Thus, in detail we find

$$R_{\mathrm{f}}^{+}\left(\hat{k}\right)=K_{\mathrm{f}}^{+}\left(\hat{k}\right)_{\mathrm{class}}\left[1-\frac{\mathrm{i}\beta^{2}}{16\pi}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)-\frac{\mathrm{i}\beta^{2}}{8\pi}\frac{\theta}{\cosh\theta}\right], \tag{4.2}$$

where $\hat{k}=\sqrt{\omega^{2}-m^{2}}$. This agrees precisely with Eq. (3.5) provided we take ρ_0=1.

The other fermion reflection factor, corresponding to the boundary condition $\psi_1=-\psi_2$, can be calculated to the same order in a similar manner to obtain

$$R_{\mathrm{f}}^{-}\left(\hat{k}\right)=K_{\mathrm{f}}^{-}\left(\hat{k}\right)_{\mathrm{class}}\left[1-\frac{\mathrm{i}\beta^{2}}{16\pi}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)+\frac{\mathrm{i}\beta^{2}}{8\pi}\frac{\theta}{\cosh\theta}\right], \tag{4.3}$$

This expression also agrees with the expression for the fermionic reflection factor corresponding to the boundary conditions BC^{-} which was quoted in Eq. (3.16).

We have studied the boundary fermion reflection factors for the supersymmetric sinh-Gordon model perturbatively up to one loop. It is gratifying that the results are in agreement with various conjectures obtained on general grounds.

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超对称 Sinh-Gordon 模型的边界反射因子

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摘要　利用微扰论研究了带有一个可积边界条件的 sinh-Gordon 模型,检验了精确反射因子的一圈修正, 并构造了玻色子和费米子传播子.

关键词　超对称　sinh-Gordon 模型　反射因子