$U_q(\hat{sl}_2)$的两维循环表示和八顶角 Ising 模型*

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摘 要

明显地构造了 $U_q(\hat{sl}_2)$的二维循环表示及其扭结子. 此循环表示是赋值(evaluation)表示, 而扭结子(intertwiner)则是八顶角 Ising 模型的 R 矩阵.

关键词 量子代数, 循环表示, 扭结子, Ising 模型.

q 为单位根时量子代数 $U_q(\hat{sl}_2)$的表示[1,2] 是与手征 Potts 模型相联系的[3,4,5,6]. 当 q 是 N 阶单位根且 N 为奇数时, $U_q(\hat{sl}_2)$ 的循环表示在文献 [5,7,8] 中作了讨论. 文献 [9] 证明了六顶角 Ising 模型是 $U_q(\hat{sl}_2)$取 $q^4=1$ 时的纽结子. 本文将讨论当 $q^4=1$ 时 $U_q(\hat{sl}_2)$的循环表示并将证明这个表示的扭结子给出八顶角 Ising 模型的 R 矩阵.

1 $U_q(\hat{sl}_2)$代数

首先约定一些记号. 仿射量子代数 $U_q(\hat{sl}_2)$的 Cartan 矩阵为

$$(a_{ij})=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

它由$\{e_i, f_i, K_i^{\pm}, z_i^{\pm}\}$ 生成, 具有代数关系

$$\begin{cases} [z_i, \ \cdot\ ]=0, \ \forall\ \cdot\ \in U_q(\hat{sl}_2) \\ K_iK_i^{-1}=K_i^{-1}K_i=1, \\ K_i^{l_1}K_j^{l_2}=K_j^{l_2}K_i^{l_1}, \ \forall\ l_1, l_2=\pm, \ \forall\ i,j \\ K_ie_j=q^{a_{ij}}e_jK_i, \\ K_if_j=q^{-a_{ij}}f_jK_i, \\ [e_i, f_j]=\delta_{ij}\dfrac{K_i-K_i^{-1}}{q-q^{-1}}, \\ \Sigma_{\nu=0}^3(-)^\nu\begin{bmatrix}3\\ \nu\end{bmatrix}_q e_i^{3-\nu}e_je_i^\nu=0, \\ \Sigma_{\nu=0}^3(-)^\nu\begin{bmatrix}3\\ \nu\end{bmatrix}_q f_i^{3-\nu}f_jf_i^\nu=0, \end{cases}$$

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其中中心元素 z_i 归因于量子对偶构造. 作为Hopf结构映射的余乘定义为

$$\begin{cases} \Delta(e_i) = e_i \otimes 1 + z_i K_i \otimes e_i, \\ \Delta(f_i) = f_i \otimes K_i^{-1} + z_i^{-1} \otimes f_i, \\ \Delta(K_i^{\pm}) = K_i^{\pm} \otimes K_i^{\pm}, \\ \Delta(z_i^{\pm}) = z_i^{\pm} \otimes z_i^{\pm}. \end{cases} \tag{1}$$

上述定义给出了一组有不同 q 值的代数族.

1.1 $q=i$ 时的 $U_q(\hat{sl}_2)$ 代数

以下, 我们将集中讨论 $q=i$ 这一特殊情形. 为方便计, 我们对生成元 f_i 作一同构变换, 同时引入新的中心元素 c_0 和 c_1 以利于构造扭结子. 这样, 代数关系就变成

$$\begin{cases} [z_i, \cdot] = [c_i, \cdot] = 0, \quad \forall \cdot \in U_q(\hat{sl}_2), \\ K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ K_i^{l_1} K_j^{l_2} = K_j^{l_2} K_i^{l_1}, \quad \forall\, l_1, l_2 = \pm, \quad \forall\, i, j \\ \{K_i, E_j\} = 0, \quad \forall\, i, j \\ \{K_i, F_j\} = 0, \quad \forall\, i, j \\ \{E_i, F_j\} = \delta_{ij} c_i (K_i^2 - 1), \quad \forall\, i, j \end{cases}$$

而Serre关系变为

$$[E_i^2, \{E_i, E_j\}] = [F_i^2, \{F_i, F_j\}] = 0, \quad (i \neq j) \tag{2}$$

显然 E_i^2 和 $E_j(i\neq j)$ 不对易. 同样, F_i^2 和 $F_j(i\neq j)$ 也不对易. 关系(2)意味着

$$\begin{cases} [E_i^{2m-1}, E_j] = (-)^{m-1} E_i^{m-1} [E_i, E_j] E_i^{m-1}, \quad \forall\, i \neq j \quad A{:}Mm \geqslant 1, \\ [E_i^{2m}, E_j] = (-)^{m-1} m E_i^{m-1} [E_i^2, E_j] E_i^{m-1}, \quad \forall\, i \neq j \quad A{:}Mm \geqslant 1, \\ [F_i^{2m-1}, F_j] = (-)^{m-1} F_i^{m-1} [F_i, F_j] F_i^{m-1}, \quad \forall\, i \neq j \quad A{:}Mm \geqslant 1, \\ [F_i^{2m}, F_j] = (-)^{m-1} m F_i^{m-1} [F_i^2, F_j] F_i^{m-1}, \quad \forall\, i \neq j \quad A{:}Mm \geqslant 1. \end{cases}$$

1.2 商代数 U_q

下面我们取代数的商, 使得

$$[E_i^2, \cdot] = [F_i^2, \cdot] = 0, \quad \forall \cdot \in U_q.$$

这样就得到一新代数 U_q, 其中心子代数为

$$Z = \text{gen}\{1, z_i, c_i, E_i^2, F_i^2, K_i^2, i = 0, 1\}.$$

可以看出, 在双代数水平上, Z 仍为中心子代数.

2 U_q 及其循环表示

为了构造二维循环表示, 引入由算符 X 和 Z 生成的Weyl代数, 满足下列关系

$$X^2 = Z^2 = 1, \quad ZX = -XZ.$$

其矩阵表示为

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

重新参数化后此代数的Weyl实现如下

$$\begin{cases} F_1 = i(-XZy_0 + X\mu_1\mu_0 x_1)/(2a_0 a_1 \mu_1 \mu_0 x_1 y_1 k_1), \\ F_0 = i(Xy_1 - ZX\mu_1\mu_0 x_0)/(2a_0 a_1 x_1 y_1 k_0), \\ E_1 = i k_1 (Xy_1 - ZX\mu_1\mu_0 x_0)/2, \\ E_0 = i k_0 (-XZy_0 + X\mu_1\mu_0 x_1)/(2\mu_1\mu_0), \\ K_1 = ia_1 Z/a_0, \\ K_0 = -ia_0 Z/a_1, \\ z_1 = (a_0 a_1 y_1)/(x_0 i)\cdot 1, \\ z_0 = (x_0 i)/(a_0 a_1 y_1)\cdot 1, \\ c_0 = a_1(x_1 y_1 + x_0 y_0)[2a_0 x_1 y_1 (a_1^2 + a_0^2)]^{-1}, \\ c_1 = a_0(x_1 y_1 + x_0 y_0)[2a_1 x_1 y_1 (a_1^2 + a_0^2)]^{-1}, \end{cases}$$

其中

$$a_0 = \sqrt{\frac{y_0}{x_1\mu_1\mu_0}}, \quad a_1 = \sqrt{\frac{x_0\mu_1\mu_0}{y_1}}.$$

应当强调此表示是 $K_0 K_1 = 1$ 时的赋值表示，且

$$E_1 = (k_1 k_0 a_1 a_0 x_1 y_1)F_0, \quad F_1 = \frac{k_0}{a_1 a_0 x_1 y_1 k_1} E_0.$$

3 余　　乘

对每个生成元的余乘定义如下

$$\begin{cases} \Delta(E_i) = E_i \otimes 1 + z_i K_i \otimes E_i, \\ \Delta(F_i) = F_i \otimes 1 + z_i^{-1} K_i \otimes F_i, \\ \Delta(K_i) = K_i \otimes K_i, \\ \Delta(z_i) = z_i \otimes z_i, \\ \Delta(c_i) = \dfrac{c_i K_i^2 \otimes 1 - c_i \otimes 1 + K_i^2 \otimes K_i^2 c_i - K_i^2 \otimes c_i}{K_i^2 \otimes K_i^2 - 1 \otimes 1}. \end{cases}$$

生成元余乘的表达式为

$$\begin{cases} \Delta E_1 = X_1 y_{11} + Z_1 X_2 \mu_{11}\mu_{10} y_{21} - Z_1 X_1 \mu_{11}\mu_{10} x_{10} - Z_1 Z_2 X_2 \mu_{21}\mu_{20}\mu_{11}\mu_{10} x_{20} \\ \Delta E_0 = X_1 \mu_{21}\mu_{20}\mu_{11}\mu_{10} x_{11} + Z_1 X_2 \mu_{21}\mu_{20} x_{21} + Z_1 X_1 \mu_{21}\mu_{20} y_{10} + Z_1 Z_2 X_2 y_{20} \\ \Delta F_0 = X_2 y_{21} y_{11} y_{10} - X_1 Z_2 \mu_{21}\mu_{20} x_{21} x_{20} y_{11} - Z_2 X_2 \mu_{21}\mu_{20} x_{20} y_{11} y_{10} \\ \qquad + Z_1 X_1 Z_2 \mu_{21}\mu_{20}\mu_{11}\mu_{10} x_{21} x_{20} x_{10} \\ \Delta F_1 = -X_2 \mu_{21}\mu_{20}\mu_{11}\mu_{10} x_{21} x_{11} x_{10} + X_1 Z_2 \mu_{11}\mu_{10} x_{11} y_{21} y_{20} \\ \qquad - Z_2 X_2 \mu_{11}\mu_{10} x_{11} x_{10} y_{20} + Z_1 X_1 Z_2 y_{21} y_{20} y_{10} \end{cases}$$

其中 $Z_1 = Z\otimes 1$，$Z_2 = 1\otimes Z$，对 X_1 和 X_2 也有类似式子．注意，在上述表达式中已略去了整体系数，它们在下面扭结子给出的相似变换作用下保持不变，因而与扭结子的构造没有关系．

4 扭 结 子

4.1 构造

代数的张量表示和扭结子座落在四个因子曲线上，即，

曲线 C_{10} 由 x_{10}，y_{10}，μ_{10} 描述；曲线 C_{11} 由 x_{11}，y_{11}，μ_1 描述；曲线 C_{20} 由 x_{20}，y_{20}，μ_{20} 描述；曲线 C_{21} 由 x_{21}，y_{21}，μ_{21} 描述.

张量表示是在乘积曲线 $U=U_{10,11}\times U_{20,21}$ 上的，这里 $U_{10,11}=C_{10}\times C_{11}$，而 $U_{20,21}=C_{20}\times C_{21}$．因此 $U=(C_{10}\times C_{11})\times(C_{20}\times C_{21})$．曲线方程为

$$y_{ij}^2=\frac{x_{ij}^2-k}{kx_{ij}^2-1}\ ,\quad \mu_{ij}^2=\frac{k'}{1-kx_{ij}^2}\ ,$$

其中 $k'^2=1-k^2$ 和 k，k'是共轭的椭圆模参量．可以看到，对曲线 C_{ij} 而言，i 代表表示空间 ($i=1,2$)，而 j 是仿射指标 ($j=0,1$).

定义在四个因子曲线上的扭结子也可分解成四个因子，即四个局域扭结子，定义如下

(1) 局域扭结子 (I)：曲线 C_{10} 和 C_{21} 之间的扭结 (简记为 $1,0\Longleftrightarrow 2,1$)

$$S_{10,21}=1+s_1X_1X_2\ ,\quad s_1=\frac{\mu_{2,1}x_{2,1}+\mu_{1,0}x_{1,0}}{\mu_{2,1}y_{1,0}-\mu_{1,0}y_{2,1}}\ ,$$

(2) 局域扭结子 (II)：$1,0\Longleftrightarrow 2,0$

$$T_{10,20}=1+t_2Z_2\ ,\quad t_2=\frac{y_{2,0}+y_{1,0}}{\mu_{1,0}\mu_{2,0}(x_{1,0}-x_{2,0})}\ ,$$

(3) 局域扭结子 (III)：$1,1\Longleftrightarrow 2,1$

$$T_{11,21}=1+t_1Z_1\ ,\quad t_1=\frac{y_{2,1}+y_{1,1}}{\mu_{1,1}\mu_{2,1}(x_{1,1}-x_{2,1})}\ ,$$

(4) 局域扭结子 (IV)：$1,1\Longleftrightarrow 2,0$

$$S_{11,20}=1+s_2X_1X_2\ ,\quad s_2=\frac{-\mu_{2,0}x_{2,0}-\mu_{1,1}x_{1,1}}{-\mu_{2,0}y_{1,1}+\mu_{1,1}y_{2,0}}\ .$$

整体扭结子为

$$R(1,2)=S_{11,20}T_{10,20}T_{11,21}S_{10,21}\ . \tag{3}$$

我们依次列出每个局域扭结子 $S_{10,21}$，$T_{11,21}$，$T_{10,20}$ 和 $S_{11,20}$ 连续作用的效果

(I) E_1 的余乘ΔE_1 依次映射到 ΔE_1^{I}，ΔE_1^{II}，$\Delta E_1^{\mathrm{III}}$ 最后到 ΔE_1^{IV}：

$$\begin{cases}\Delta E_1=X_1y_{11}+Z_1X_2\mu_{11}\mu_{10}y_{21}-Z_1X_1\mu_{11}\mu_{10}x_{10}-Z_1Z_2X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{20}\ ,\\ \Delta E_1^{\mathrm{I}}=X_1y_{11}-Z_1X_2\mu_{11}\mu_{21}y_{10}+Z_1X_1\mu_{11}\mu_{21}x_{21}-Z_1Z_2X_2\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{20}\ ,\\ \Delta E_1^{\mathrm{II}}=X_1y_{11}-Z_1X_2\mu_{11}\mu_{21}y_{20}+Z_1X_1\mu_{11}\mu_{21}x_{21}-Z_1Z_2X_2\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{10}\ ,\\ \Delta E_1^{\mathrm{III}}=X_1y_{21}-Z_1X_2\mu_{11}\mu_{21}y_{20}+Z_1X_1\mu_{11}\mu_{21}x_{11}-Z_1Z_2X_2\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{10}\ ,\\ \Delta E_1^{\mathrm{IV}}=X_1y_{21}+Z_1X_2\mu_{20}\mu_{21}y_{11}-Z_1X_1\mu_{20}\mu_{21}x_{20}-Z_1Z_2X_2\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{10}\ .\end{cases}$$

(II) E_0 的余乘映射成像:

$$\begin{cases}\Delta E_0 = X_1\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{11} + Z_1X_2\mu_{21}\mu_{20}x_{21} + Z_1X_1\mu_{21}\mu_{20}y_{10} + Z_1Z_2X_2y_{20}, \\ \Delta E_0^{\mathrm{I}} = X_1\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{11} - Z_1X_2\mu_{10}\mu_{20}x_{10} - Z_1X_1\mu_{10}\mu_{20}y_{21} + Z_1Z_2X_2y_{20}, \\ \Delta E_0^{\mathrm{II}} = X_1\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{11} - Z_1X_2\mu_{10}\mu_{20}x_{20} - Z_1X_1\mu_{10}\mu_{20}y_{21} + Z_1Z_2X_2y_{10}, \\ \Delta E_0^{\mathrm{III}} = X_1\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{21} - Z_1X_2\mu_{10}\mu_{20}x_{20} - Z_1X_1\mu_{10}\mu_{20}y_{11} + Z_1Z_2X_2y_{10}, \\ \Delta E_0^{\mathrm{IV}} = X_1\mu_{10}\mu_{20}\mu_{11}\mu_{21}x_{21} + Z_1X_2\mu_{10}\mu_{11}x_{11} + Z_1X_1\mu_{10}\mu_{11}y_{20} + Z_1Z_2X_2y_{10}.\end{cases}$$

(III) F_0 的余乘映射像为:

$$\begin{cases}\Delta F_0 = X_2y_{21}y_{11}y_{10} - X_1Z_2\mu_{21}\mu_{20}x_{21}x_{20}y_{11} - Z_2X_2\mu_{21}\mu_{20}x_{20}y_{11}y_{10} \\ \qquad + Z_1X_1Z_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{20}x_{10}, \\ \Delta F_0^{\mathrm{I}} = X_2y_{21}y_{11}y_{10} + X_1Z_2\mu_{10}\mu_{20}x_{10}x_{20}y_{11} + Z_2X_2\mu_{10}\mu_{20}x_{20}y_{11}y_{21} \\ \qquad + Z_1X_1Z_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{20}x_{10}, \\ \Delta F_0^{\mathrm{II}} = X_2y_{21}y_{11}y_{20} + X_1Z_2\mu_{10}\mu_{20}x_{10}x_{20}y_{11} + Z_2X_2\mu_{10}\mu_{20}x_{10}y_{11}y_{21} \\ \qquad + Z_1X_1Z_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{20}x_{10}, \\ \Delta F_0^{\mathrm{III}} = X_2y_{21}y_{11}y_{20} + X_1Z_2\mu_{10}\mu_{20}x_{10}x_{20}y_{21} + Z_2X_2\mu_{10}\mu_{20}x_{10}y_{11}y_{21} \\ \qquad + Z_1X_1Z_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{11}x_{20}x_{10}, \\ \Delta F_0^{\mathrm{IV}} = X_2y_{21}y_{11}y_{20} - X_1Z_2\mu_{10}\mu_{11}x_{10}x_{11}y_{21} - Z_2X_2\mu_{10}\mu_{11}x_{10}y_{20}y_{21} \\ \qquad + Z_1X_1Z_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{11}x_{20}x_{10}.\end{cases}$$

(IV) 而 F_1 的余乘则映射成:

$$\begin{cases}\Delta F_1 = -X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{11}x_{10} + X_1Z_2\mu_{11}\mu_{10}x_{11}y_{21}y_{20} \\ \qquad - Z_2X_2\mu_{11}\mu_{10}x_{11}x_{10}y_{20} + Z_1X_1Z_2y_{21}y_{20}y_{10}, \\ \Delta F_1^{\mathrm{I}} = -X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{11}x_{10} - X_1Z_2\mu_{11}\mu_{21}x_{11}y_{10}y_{20} \\ \qquad + Z_2X_2\mu_{11}\mu_{21}x_{11}x_{21}y_{20} + Z_1X_1Z_2y_{21}y_{20}y_{10}, \\ \Delta F_1^{\mathrm{II}} = -X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{11}x_{20} - X_1Z_2\mu_{11}\mu_{21}x_{11}y_{10}y_{20} \\ \qquad + Z_2X_2\mu_{11}\mu_{21}x_{11}x_{21}y_{10} + Z_1X_1Z_2y_{21}y_{20}y_{10}, \\ \Delta F_1^{\mathrm{III}} = -X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{11}x_{20} - X_1Z_2\mu_{11}\mu_{21}x_{21}y_{10}y_{20} \\ \qquad + Z_2X_2\mu_{11}\mu_{21}x_{11}x_{21}y_{10} + Z_1X_1Z_2y_{11}y_{20}y_{10}, \\ \Delta F_1^{\mathrm{IV}} = -X_2\mu_{21}\mu_{20}\mu_{11}\mu_{10}x_{21}x_{11}x_{20} + X_1Z_2\mu_{20}\mu_{21}x_{21}y_{10}y_{11} \\ \qquad - Z_2X_2\mu_{20}\mu_{21}x_{20}x_{21}y_{10} + Z_1X_1Z_2y_{11}y_{20}y_{10}.\end{cases}$$

4.2　扭结子的明显表达式

扭结子的非零矩阵元为

$$\begin{aligned} R_{1,1} &= t_1t_2s_2s_1 + t_1t_2 - t_1s_2s_1 + t_1 - t_2s_2s_1 + t_2 + s_2s_1 + 1, \\ R_{1,4} &= t_1t_2s_2 + t_1t_2s_1 - t_1s_2 + t_1s_1 - t_2s_2 + t_2s_1 + s_2 + s_1, \\ R_{2,2} &= -t_1t_2s_2s_1 - t_1t_2 - t_1s_2s_1 + t_1 + t_2s_2s_1 - t_2 + s_2s_1 + 1, \\ R_{2,3} &= -t_1t_2s_2 - t_1t_2s_1 - t_1s_2 + t_1s_1 + t_2s_2 - t_2s_1 + s_2 + s_1, \\ R_{3,2} &= -t_1t_2s_2 - t_1t_2s_1 + t_1s_2 - t_1s_1 - t_2s_2 + t_2s_1 + s_2 + s_1, \\ R_{3,3} &= -t_1t_2s_2s_1 - t_1t_2 + t_1s_2s_1 - t_1 - t_2s_2s_1 + t_2 + s_2s_1 + 1, \\ R_{4,1} &= t_1t_2s_2 + t_1t_2s_1 + t_1s_2 - t_1s_1 + t_2s_2 - t_2s_1 + s_2 + s_1, \\ R_{4,4} &= t_1t_2s_2s_1 + t_1t_2 + t_1s_2s_1 - t_1 + t_2s_2s_1 - t_2 + s_2s_1 + 1, \end{aligned} \tag{4}$$

4.3 自由费米子条件

可以直接验证，对 $t_{1,2}$ 和 $s_{1,2}$ 的任何值，方程 (4) 中的 R 矩阵均满足自由费米子条件[10]

$$R_{11}R_{44}+R_{23}R_{32}=R_{14}R_{41}+R_{22}R_{33}.$$

这个条件保证了由扭结子给出的 Boltzman 权构成一个可积的八顶角模型. 应当强调，正是方程 (3) 给出的扭结子的因子化，使这个条件得到满足. 下节将证明我们得到的扭结子确实给出了八顶角 (椭圆) Ising 模型的 R 矩阵.

5 八顶角Ising 模型

为了把方程 (4) 给出的扭结子参数化成椭圆函数，令

$$\begin{aligned}
x(u_{10})&=(\mathrm{sn}(u_{10})\sqrt{k})^{-1}, & x(u_{20})&=(\mathrm{sn}(u_{20})\sqrt{k})^{-1},\\
y(u_{10})&=\sqrt{k^{-1}}\ \mathrm{dc}(u_{10}), & y(u_{20})&=\sqrt{k^{-1}}\ \mathrm{dc}(u_{20}),\\
\mu(u_{10})&=\mathrm{i}\sqrt{k'}\ \mathrm{sc}(u_{10}), & \mu(u_{20})&=-\mathrm{i}\sqrt{k'}\ \mathrm{sc}(u_{20}),\\
x(u_{11})&=-\sqrt{k}\ \mathrm{sn}(u_{11}), & x(u_{21})&=-\sqrt{k}\ \mathrm{sn}(u_{21}),\\
y(u_{11})&=\sqrt{k}\ \mathrm{cd}(u_{11}), & y(u_{21})&=\sqrt{k}\ \mathrm{cd}(u_{21}),\\
\mu(u_{11})&=\sqrt{k'}\ (\mathrm{dn}(u_{11}))^{-1}, & \mu(u_{21})&=\sqrt{k'}\ (\mathrm{dn}(u_{21}))^{-1},
\end{aligned}$$

其中用了缩略记号，如 $\mathrm{sc}(x)=\dfrac{\mathrm{sn}(x)}{\mathrm{cn}(x)}$. 注意 u_{ij} ($i=1$，2 和 $j=0$，1) 是独立参数. 作为一个特殊情形，可进一步令

$$u_{10}=-u_{21}-u,\ u_{20}=-u_{21}+2\mathrm{i}K',\ u_{11}=u_{21}-2\mathrm{i}K'+u,$$

这时扭结子可表为

$$\hat{R}(u)=\frac{1+\mathrm{cd}(u)}{4}[1+(k\ \mathrm{sn}(u)-\mathrm{i}\ \mathrm{dn}(u))X\otimes X]\left(1+\frac{\mathrm{dn}(u)-\mathrm{cn}(u)}{k'\,\mathrm{sn}(u)}Z\right)$$
$$\otimes\left(1-\frac{k'\,\mathrm{sn}(u)}{\mathrm{dn}(u)+\mathrm{cn}(u)}Z\right)[1+(k\ \mathrm{sn}(u)+\mathrm{i}\ \mathrm{dn}(u))X\otimes X],$$

这正好就是八顶角Ising 模型的 R 矩阵

$$\hat{R}(u)=\begin{bmatrix}\mathrm{cd}(u) & 0 & 0 & k\ \mathrm{sn}(u)\mathrm{cd}(u)\\ 0 & 1 & \mathrm{sn}(u)g & 0\\ 0 & \mathrm{sn}(u)g^{-1} & 1 & 0\\ k\ \mathrm{sn}(u)\mathrm{cd}(u) & 0 & 0 & \mathrm{cd}(u)\end{bmatrix},$$

其中 g 是规范变换参数

$$g=-k-\mathrm{i}k'.$$

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Two Dimensional Cyclic Representations of $U_q(\hat{sl}_2)$ and 8－Vertex Ising Model

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Abstract

We give explicitly the 2－dimensional cyclic representations of quantum algebra $U_q(\hat{sl}_2)$ with central extension. The intertwiner for tensor representations in different ordersis constructed with C－G coefficients. This intertwiner is shown to be the R－matrix for eight vertex Ising model.

Key words　quantum algebra, cyclic representation, intertwiner, Ising model.